A primal–dual symmetric relaxation for homogeneoustric systems

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Abstract

We address the feasibility (existence of non-trivial solutions) of the pair of alternative conic systems of constraints

\[ Ax = 0, \quad x \in C \]

and

\[ -A^T y \in C^*, \]

where \( A \in \mathbb{R}^{m \times n} \), \( m < n \), and \( C \subseteq \mathbb{R}^n \) is a closed convex cone. To this end, we reformulate the above pair of conic systems as a primal-dual pair of conic programs. Each of the conic programs corresponds to a natural relaxation of each of the two conic systems.

When \( C \) is a self-scaled cone with a known self-scaled barrier, the conic programming reformulation can be solved via an interior-point algorithm. For a well-posed instance \( A \), a strict solution to one of the two original conic systems can be obtained in \( O(\sqrt{\nu_C} \log(\nu_C C(A))) \) interior-point iterations. Here \( \nu_C \) is the complexity parameter of the self-scaled barrier of \( C \) and \( C(A) \) is Renegar’s condition number of \( A \). A central feature of our approach is the conditioning of the system of equations that arise at each interior-point iteration. The condition number of such system of equations grows in a controlled manner and remains bounded by a constant factor of \( C(A)^2 \) throughout the entire algorithm.

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1 Introduction

We study the conic feasibility systems of constraints

\[ Ax = 0, \quad x \in C \quad (1) \]

and

\[ -A^T y \in C^*, \quad (2) \]

where \( A \in \mathbb{R}^{m \times n} \) and \( C \subseteq \mathbb{R}^n \) is a closed convex cone.

The conic systems (1), (2) are essentially alternative systems: either system is well-posed feasible if and only if the other does not have non-zero solutions.

We reformulate the above pair of feasibility problems as a primal-dual pair of conic programs. The conic programs correspond to natural relaxations of the original conic systems. When the cone \( C \) is self-scaled, the primal-dual reformulation can be solved via an interior-point algorithm. For a well-posed instance \( A \), the algorithm determines which one of the conic systems is feasible and generates a strictly feasible solution.

A key feature of our approach is that the amount of computational work needed to solve the feasibility pair (1), (2) depends naturally on Renegar’s condition number of the data instance \( A \): The algorithm that we propose solves the above feasibility problem in a number of interior-point iterations that is proportional to \( \mathcal{C}(A) \) (see Theorem 3.1). In addition, the condition number of the equations arising at each interior-point iteration is controlled throughout the algorithm and is bounded by a factor of \( \mathcal{C}(A) \) (see Theorem 4.7). The first property, i.e., the complexity dependence on \( \mathcal{C}(A) \) is in the same spirit as condition-based approaches to optimization and conic feasibility problems such as [5, 6, 7, 16, 17]. Indeed, our approach extends [16], where a purely “primal” approach for the feasibility problem \( Ax = 0, x \in C \) was presented and analyzed in terms of \( \mathcal{C}(A) \). The second property, i.e., the control of the condition numbers of the equations arising at each main iteration, provides a central foundational step for a rigorous study of the behavior of algorithms that perform finite precision arithmetic computations. Results of this type have been scarce in optimization, except for a few papers that address this issue in the linear programming case [2, 4, 21]. In all of these cases, a bound on the condition number of the equations arising at each main iteration is the crucial element at the heart of the roundoff analysis. The controlled conditioning of the equations arising in our approach paves the way for a formal study of the effects of finite-precision arithmetic for more general optimization problems. Indeed, in [3] we extend the scope of [2] to study the solvability of second-order cone feasibility problems under finite precision.

Some of our ideas are inspired by previous work by Peña and Renegar [18], and by Cucker and Peña [2]. In the former, a purely primal relaxation scheme for solving (1) was proposed and analyzed. In the latter, the authors devised a primal-dual scheme for solving (1) or (2) for the particular case when \( C = \mathbb{R}_+^n \),
using a finite precision machine. This paper combines and extends both of these previous works. Unlike the purely primal approach used in [16], we reformulate the feasibility problem as a primal-dual pair of conic programs, in the same spirit as [2] but for more general conic systems. As a nice consequence of this primal-dual approach, both (1) and (2) are treated in a unified manner, without any a priori feasibility assumption of either system. The proofs presented herein are not only more general, but also more concise and transparent than the analogous ones in [2, 16]. Our main results rely on a general perturbation theorem for self-scaled programs (Theorem 5.1), which is of independent interest. To ease our presentation, we develop the main technical ideas in the last section of the paper.

It should be mentioned that constructions based on homogenization and relaxation, as the one that underlies our work, have been known and used in optimization. For instance, Vavasis and Ye [20] propose a relaxation and homogenization scheme to solve a polyhedral systems of constraints. Ye, Todd, and Mizuno [22] give a homogeneous self-dual formulation to solve linear programs when no a priori feasibility information is available. Nesterov, Todd, and Ye [12] propose yet a more general self-dual formulation that either detects infeasibility or finds near-optimal solutions for more general conic programs. However, the two central properties of our approach, namely the natural analysis of the complexity of the solution to (1), (2) in terms of its measure of well-posedness $C(A)$, as well as the controlled conditioning of the equations arising at each main iteration have not been previously developed.

The paper has been organized as follows. Section 2 reviews Renegar’s distance to ill-posedness and condition number for conic systems, as well as some basic facts about barrier and self-scaled barrier functions. Section 3 develops our central construction: we recast the pair (1), (2) as the primal-dual pair of conic programs (9), (10). In Section 4 we describe an interior-point algorithm that computes a strict solution to whichever of (1), (2) is feasible, provided the data $A$ is well-posed. The algorithm finds such a solution within $O(\sqrt{C} \log(\nu_C C(A)))$ interior-point iterations, where $\nu_C$ is the complexity parameter of barrier for the cone $C$, and $C(A)$ is the condition number of the data instance $A$. In addition, we study the system of equations arising at each interior-point iteration, whose solution constitutes the bulk of the computational work at each interior-point iteration. We show that the condition number of such system of equations remains controlled throughout the algorithm, and is bounded above by a factor of $C(A)$ (see Theorem 4.7). Thus, for well-posed instances, the algorithm will solve the feasibility problem in a small number of iterations and will only need to solve well-conditioned systems of equations. Furthermore, at any intermediate iteration before termination the algorithm automatically yields a lower bound on the condition number $C(A)$. In other words, when the input matrix $A$ is poorly conditioned, our algorithm will provide a “certificate of bad-conditioning” as long as the feasibility problem remains unsolved. In the last section we develop the main technicalities of the paper. Section 5.1 presents Theorem 5.1, a perturbation result for self-scaled programs. Theorem 5.1 is subsequently used to prove the central results in Sections 4, namely Theorem 4.7, Propositions 4.4 and 4.5.
2 Preliminaries

2.1 Renegar’s distance to ill-posedness and condition number

We next review the basic definitions and properties of Renegar’s condition number and distance to ill-posedness (see [13, 15] for a detailed discussion on these concepts). We say that (1) is a well-posed feasible system if

\[ \{ Ax : x \in C \} = \mathbb{R}^m. \]  

(3)

Let \( P \) be the set of \( m \times n \) matrices \( A \) such that (3) holds. Notice that \( A \in P \) if and only if the alternative system (2) does not have nonzero solutions.

We say that (2) is a well-posed feasible system if

\[ \{ A^T y : y \in \mathbb{R}^m \} + C^* = \mathbb{R}^n. \]  

(4)

Let \( D \) be the set of \( m \times n \) matrices \( A \) such that (4) holds. Notice that \( A \in D \) if and only if the alternative system (1) does not have nonzero solutions.

Throughout the paper, we assume that each Euclidean space \( \mathbb{R}^d \) is endowed with a fixed inner product. For any given vector \( v \in \mathbb{R}^d \), we will write \( \|v\| \) to denote its Euclidean norm. Likewise, for any given matrix \( M \in \mathbb{R}^{d \times p} \), we will write \( \|M\| \) to indicate its operator norm, that is \( \|M\| := \max\{\|Mv\| : \|v\| = 1\} \).

It can be shown that both \( P \) and \( D \) are open subsets of \( \mathbb{R}^{m \times n} \). The set \( \mathbb{R}^{m \times n} \setminus (P \cup D) \) is the set of ill-posed instances. It is easy to show that this set has Lebesgue measure equal to zero. Furthermore, if \( m < n \) and \( C \) is a regular cone (i.e., both \( C \) and \( C^* \) have nonempty interior), then the closure of either \( P \) or \( D \) in \( \mathbb{R}^{m \times n} \) is the complement of the other.

The distance to infeasibility of (1) is defined as

\[ \rho_P(A) := \inf\{\|\Delta A\| : A + \Delta A \notin P\}. \]

Likewise, the distance to infeasibility of (2) is defined as

\[ \rho_D(A) := \inf\{\|\Delta A\| : A + \Delta A \notin D\}. \]

The distance to ill-posedness of \( A \) is

\[ \rho(A) := \inf\{\|\Delta A\| : A + \Delta A \notin P \cup D\} = \max\{\rho_P(A), \rho_D(A)\}. \]

The data instance \( A \) is well-posed if \( \rho(A) > 0 \), i.e., if \( A \in P \cup D \). Renegar’s condition number \( C(A) \) is defined as the reciprocal of the relative distance to ill-posedness, i.e.,

\[ C(A) := \frac{\|A\|}{\rho(A)}. \]

Our treatment will rely on the following characterizations of the distance to infeasibility. For a detailed discussion on this and closely related issues see [13, 15, 17].
Proposition 2.1 (Renegar) For any given $A \in \mathbb{R}^{m \times n}$,
\[
\rho_P(A) = \sup \{ \delta : \|v\| \leq \delta \Rightarrow v \in \{Ax : \|x\| \leq 1, x \in C\}\},
\]
and
\[
\rho_D(A) = \sup \{ \delta : \|u\| \leq \delta \Rightarrow u \in \{A^Ty : \|y\| \leq 1\} + C^*\}.
\]

2.2 Self-scaled barrier functions

Given a self-scaled barrier function $f$, we shall use $g$ and $H$ to denote the gradient and Hessian of $f$ respectively. We shall also let $\nu_f$ denote the barrier parameter of $f$. In such case we shall say that $f$ is a $\nu_f$-self-scaled barrier function. We shall also let $D_f$ denote the domain of $f$.

Some of our statements are phrased in terms of the local inner product and local norm, which we now recall. Given a barrier function $f$ and a point $x \in D_f$, the local inner product $\langle \cdot, \cdot \rangle_x$ induced by $x$ is defined as
\[
\langle u, v \rangle_x := \langle u, H(x)v \rangle.
\]
The local inner norm $\| \cdot \|_x$ is defined as
\[
\|v\|_x := \langle v, v \rangle_x^{1/2}.
\]
We let $B_x$ denote the local unit ball $\{v : \|v\|_x \leq 1\}$.

Our development relies on the following key properties of self-scaled barrier functions. (For a detailed discussion on these properties see \[9, 10, 11, 18\].)

Proposition 2.2 Let $f$ be a $\nu$-self-scaled barrier function and $x \in D_f$.

(a) If $\|y - x\|_x < 1$ then $y \in D_f$ and for all $v \neq 0$
\[
1 - \|y - x\|_x \leq \frac{\|v\|_y}{\|v\|_x} \leq \frac{1}{1 - \|y - x\|_x}.
\]

(b) $\{z \in D_f : \langle z - x, g(x) \rangle \geq 0\} \subseteq \{z : \|z - x\|_x \leq \nu\}$.

(c) $-g(x) \in D_f$, $-g(-g(x)) = x$, $H(-g(x)) = H(x)^{-1}$, and $\|H(x)^{-1}\| \leq \|x\|^2$.

(d) For $\alpha > 0$
\[
g(\alpha x) = \frac{1}{\alpha} g(x), \quad \text{and} \quad H(\alpha x) = \frac{1}{\alpha^2} H(x).
\]

(e) $\langle x, g(x) \rangle = -\nu$.

(f) Given a point $s \in D_f$, there exists a unique scaling point $w \in D_f$ such that
\[
H(w)x = s, \quad \text{and} \quad H(w)g(s) = g(x).
\]

Furthermore, for $\mu > 0$ the point $\bar{w} := \sqrt{\mu}w$ satisfies
\[
\|s + \mu g(x)\|_{-\mu g(x)} \geq \min \left\{ \frac{1}{5}, \frac{4}{5}\|s - \bar{w}\|_{\bar{w}} \right\}.
\]
3 Reformulation

The following reformulation scheme is a generalization of the reformulations proposed in [2] and [16]. Recast (1) as
\[ \min \|x''\| \]
\[ \text{s.t.} \quad Ax + x'' = 0 \]
\[ x \in C \]
\[ \|x\| \leq 1, \]
and recast (2) as
\[ \min \|y'\| \]
\[ \text{s.t.} \quad -A^Ty + y' \in C^* \]
\[ \|y\| \leq 1. \]

By introducing auxiliary variables, the problems (5) and (6) are equivalent to the pair
\[ \min \tau \]
\[ \text{s.t.} \quad Ax + x'' = 0 \]
\[ -x + x' = 0 \]
\[ t = 1 \]
\[ x \in C \]
\[ \|x'\| \leq t \]
\[ \|x''\| \leq \tau, \]
and
\[ \min \eta \]
\[ \text{s.t.} \quad -A^Ty + y' \in C^* \]
\[ \|y\| \leq 1 \]
\[ \|y'\| \leq \eta. \]

When cast appropriately, the pair (7), (8) is a primal-dual pair of conic programs in a higher dimensional space: Indeed, let \( K := C \times \mathbb{L}_n \times \mathbb{L}_m \), where \( \mathbb{L}_n, \mathbb{L}_m \) are second-order cones in \( \mathbb{R}^{n+1}, \mathbb{R}^{m+1} \) respectively. The problem (7) can be written as
\[ \min \langle \tilde{c}, \tilde{x} \rangle \]
\[ \text{s.t.} \quad A\tilde{x} = \tilde{b} \]
\[ \tilde{x} \in K, \]
where \( \tilde{x} = (x, t, x', x'') \in \mathbb{R}^{(m+2n+2)}, \) and \( A \in \mathbb{R}^{(m+n+1) \times (m+2n+2)}, \) \( \tilde{c} \in \mathbb{R}^{m+2n+2}, \) \( \tilde{b} \in \mathbb{R}^{m+n+1} \) are as follows
\[ A := \begin{bmatrix} A & 0 & 0 & 0 & I_m \\ 0 & 1 & 0 & 0 & 0 \\ -I_n & 0 & I_n & 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]
Now the problem (8) corresponds precisely to the dual of (9), namely
\[
\max \langle \vec{b}, \vec{y} \rangle \\
\text{s.t.} \quad A^T \vec{y} + \vec{s} = \vec{c} \\
\vec{s} \in K^*.
\] (10)

where \( \vec{y} = (y, -\eta, y') \in \mathbb{R}^{(m+n+1)} \).

It is obvious that at the optimal solutions to (9), (10) the variables \( x'' \) and \( y' \) are zero. It is also intuitively clear that an interior-point algorithm applied to (9), (10) would yield, in the limit, a strict solution for whichever of (1), (2) that is strictly feasible. One of the main goals of this paper is to formalize and make this idea more precise. Specifically, two of our key results, namely Propositions 4.4 and 4.5 below, establish a close connection between the central path of the primal-dual pair (9), (10) and the pair (1), (2) when \( C \) is self-scaled.

For the remainder of the paper we shall assume that \( C \) is a self-scaled cone with self-scaled barrier \( f_C \). It readily follows that the cone \( K \) is self-scaled as well with self-scaled barrier
\[
f_K(x, x', t, x'', \tau) = f_C(x) - \ln(t^2 - \|x'\|^2) - \ln(\tau^2 - \|x''\|^2).
\]

We also recall that, as an immediate consequence of self-scaledness, both \( C \) and \( K \) are self-dual, that is, \( C^* = C \) and \( K^* = K \).

Recall that for a given barrier function \( f \), we shall use \( g \) and \( H \) to denote the gradient and Hessian of \( f \) respectively. We will add the subindex \( C \) or \( K \) when we refer to the gradient and Hessian of \( f_C \) or \( f_K \). We shall also abbreviate \( \nu_{f_C} \) as \( \nu_C \), and \( \nu_{f_K} \) as \( \nu_K \). By construction, we have \( \nu_K = \nu_C + 4 \).

We can now state one of the central results of the paper.

**Theorem 3.1** Assume \( A \in \mathbb{R}^{m \times n} \) with \( C(A) < \infty \) is given. A suitable short-step primal-dual interior-point algorithm applied to (9), (10) halts in at most \( O(\sqrt{\nu_C} \log(\nu_C C(A))) \) interior-point iterations, yielding a strict solution to either (1) or (2).

**Proof.** See Section 4.4 \( \square \)

### 4 Solving the conic pair via a primal-dual algorithm

Propositions 4.4 and 4.5 below formalize the intuitively clear fact that points on a suitable neighborhood of the central path of (9), (10) eventually yield solutions for whichever of (1), (2) that has strictly feasible solutions. First we recall the definition of the central path, and a neighborhood of the central path that is suitable for our purposes. (For further details see 10 11 18.)
4.1 The central path

Definition 4.1 The central path of (9), (10) is the set of solutions of the nonlinear system of equations

\[
\begin{align*}
A\bar{x} &= \bar{b} \\
A^T\bar{y} + \bar{s} &= \bar{c} \\
\bar{s} + \mu g_K(\bar{x}) &= 0 \\
\bar{x}, \bar{s} &\in \text{int}(K),
\end{align*}
\]

for all values of \(\mu > 0\).

The pair (9), (10) is amenable to the machinery of primal-dual interior-point methods for self-scaled cones (cf. [10, 11, 18, 19]). There are a number of different algorithms whose specific updates depend on the choice of a particular neighborhood of the central path. For our purposes we shall use the local neighborhood \(N_\beta\) of the central path as defined next [11, 18].

Definition 4.2 Let \(\beta \in (0, 1/2)\) be given. The local neighborhood \(N_\beta\) of (9), (10) is defined as the set of points \((\bar{x}, \bar{y}, \bar{s})\) with \(\bar{x}, \bar{s} \in \text{int}(K)\) such that the following constraints hold

\[
\begin{align*}
A\bar{x} &= \bar{b} \\
A^T\bar{y} + \bar{s} &= \bar{c} \\
\|\bar{s} + \mu(\bar{x}, \bar{s})g_K(\bar{x})\|_{-g_K(\bar{x})} &\leq \beta \mu(\bar{x}, \bar{s}),
\end{align*}
\]

where \(\mu(\bar{x}, \bar{s}) := \langle \bar{x}, \bar{s} \rangle / \nu_K\).

Remark 4.3

(a) Since \(K\) is self-scaled,

\[
\|\bar{s} + \mu(\bar{x}, \bar{s})g_K(\bar{x})\|_{-g_K(\bar{x})} = \|\bar{x} + \mu(\bar{x}, \bar{s})g_K(\bar{s})\|_{-g_K(\bar{s})},
\]

for \(\bar{x}, \bar{s} \in \text{int}(K)\). Hence either of these expressions can be used in the last inequality in Definition 4.2.

(b) We will rely on the following consequence of Proposition 2.2(f): Suppose \(\beta < 1/5\) and \((\bar{x}, \bar{y}, \bar{s}) \in N_\beta\). Let \(w\) be the scaling point of \(\bar{x}, \bar{s} \in \text{int}(K)\), and put \(\bar{w} = \sqrt{\mu(\bar{x}, \bar{s})}w\). Then

\[
\|\bar{x} - \bar{w}\|_{\bar{w}} \leq \frac{5}{4}\beta.
\]

4.2 Properties of the central neighborhood

Proposition 4.4 Assume \(A \in \mathcal{P}\). Let \((\bar{x}, \bar{y}, \bar{s}) \in N_\beta\) with \(\bar{c}^T\bar{x} = \tau\). If

\[
\tau < \frac{(1 - \beta)\rho_P(A)}{\sqrt{2}(\nu_K + \beta)},
\]

then...
then
\[ \sigma_{\min}(AH_C(x)^{-1}A^T) > \frac{(1 - \beta)^2 \rho_P(A)^2}{2(\nu_K + \beta)^2} > \tau^2. \]

The latter in turn implies that the point
\[ \bar{x} := x + H_C(x)^{-1}A^T(AH_C(x)^{-1}A^T)^{-1}x'' \]
satisfies
\[ A\bar{x} = 0, \bar{x} \in \text{int}(C), \text{ and } \|\bar{x} - x\|_x \leq \frac{\sqrt{2}(\nu_K + \beta)\tau}{(1 - \beta)\rho_P(A)}. \]

Proof. See Section 5.3 \hfill \Box

4.5 Proposition Assume \( A \in \mathcal{D} \). Let \((\bar{x}, \bar{y}, \bar{s}) \in \mathcal{N}_\beta\) with \( \bar{b}^T \bar{y} = -\eta \). If \( \eta < \frac{(1 - \beta)\rho_D(A)}{\nu_K} \), then \( y \) satisfies \(-A^Ty \in \text{int}(C)\).

Proof. See Section 5.4 \hfill \Box

4.3 Initial point

Let \( e \) be the unique point in \( C \) such that \( H_C(e) = I \). The existence of such point follows from Proposition 2.2(f). Furthermore, for the basic cones that most commonly arise in practice, that is, the nonnegative orthant \( \mathbb{R}_n^+ \), the cone of symmetric positive semidefinite matrices \( \mathbb{S}_n^+ \), and the second-order cone \( \mathbb{L}_n \), this point is readily available. In each of these cases the point \( e \) is respectively
\[ [1 \ldots 1]^T, I, \text{ and } [1 \ 0 \ldots 0]^T. \]

If \( C = C_1 \times \cdots \times C_r \) is a direct product of self-scaled cones as above, then the point \( e \) for \( C \) is the concatenation \((e_1, \ldots, e_r)\), where each \( e_i \in C_i \) is the point such that \( H_{C_i}(e) = I \), for \( i = 1, \ldots, r \).

We shall assume that the point \( e \) for the cone \( C \) is available. Under this reasonable assumption it is easy to construct an initial point in \( N_\beta \). We note that \( g_C(e) = -e \) and \( \|e\| = \sqrt{\nu_C} \). (See \[10\] \[11\] \[13\].)

4.6 Proposition Let
\[ \alpha := \frac{1}{\sqrt{\nu_C + 2}}, \text{ and } M := \frac{\alpha \|Ae\|}{\beta}. \]

Then the point \((\bar{x}, \bar{y}, \bar{s})\), defined as follows, belongs to \( \mathcal{N}_\beta \):
\[ \bar{x} = (\alpha e, 1, \alpha e, 2M, -\alpha Ae) \]
\[ \bar{y} = \left(0, -\frac{M}{\alpha^2}, \frac{M}{\alpha} e\right) \]
\[ \bar{s} = \left(\frac{M}{\alpha} e, \frac{M}{\alpha^2}, -\frac{M}{\alpha} e, 1, 0\right). \]
Proof. By construction,

\[ \mu(\vec{x}, \vec{s}) = \nu_K \left( \frac{1}{\nu_K} + 2 \right) = \frac{M(\nu_K + 4)}{\nu_K} = M, \]

and

\[ g_K(\vec{x}) = \left( -\frac{1}{\alpha} e, -\frac{1}{\alpha^2} e, -2\delta M, -\delta \alpha A e \right), \]

where \( \delta := \frac{2}{4M^2 - \alpha^2 \|Ae\|^2} = \frac{2}{M^2(4 - \beta^2)}. \)

Therefore,

\[ \vec{s} + \mu(\vec{x}, \vec{s}) g_K(\vec{x}) = (0, 0, 0, 1 - 2\delta M^2, -M \delta \alpha A e). \]

Consequently,

\[
\| \vec{s} + \mu(\vec{x}, \vec{s}) g_K(\vec{x}) \|^2_{g_K(\vec{x})} = \left\langle \begin{bmatrix} 1 - 2\delta M^2 \\ -M \delta \alpha A e \end{bmatrix}, H_{\nu_K}(2M, -\alpha A e)^{-1} \begin{bmatrix} 1 - 2\delta M^2 \\ -M \delta \alpha A e \end{bmatrix} \right\rangle = \frac{1}{2} \left\langle \begin{bmatrix} 1 - 2\delta M^2 \\ -M \delta \alpha A e \end{bmatrix}, \begin{bmatrix} \alpha^2 \|Ae\|^2 \\ -2\alpha M A e \end{bmatrix} \right\rangle = \frac{1}{2} \alpha^2 \|Ae\|^2 = \frac{1}{2} \beta^2 M^2 < \beta^2 \mu(\vec{x}, \vec{s})^2.
\]

It thus follows that \((\vec{x}, \vec{y}, \vec{s}) \in \mathcal{N}_\beta\) because by construction \(A \vec{x} = \vec{b}, \begin{bmatrix} \alpha^2 \|Ae\|^2 \\ -2\alpha M A e \end{bmatrix} \vec{s} = \vec{c}, \) and \(\vec{x}, \vec{s} \in \text{int}(K). \)

\[ \square \]

4.4 The algorithm

We are now ready to describe our primal-dual algorithm. This is essentially a path-following short-step algorithm like those described in [11, Sec. 6], [18, Sec. 3.7], or [19, Sec. 3] enhanced with a specific starting point, and a stopping criterion.

The crucial step at each main iteration is the update of the iterate \((\vec{x}, \vec{y}, \vec{s})\). This is performed by putting

\[
(\vec{x}^+, \vec{y}^+, \vec{s}^+) := (\vec{x}, \vec{y}, \vec{s}) + (\Delta \vec{x}, \Delta \vec{y}, \Delta \vec{s}),
\]

where \((\Delta \vec{x}, \Delta \vec{y}, \Delta \vec{s})\) is the Nesterov-Todd direction, that is, the solution to

\[
H_K(w) \Delta \vec{x} + \Delta \vec{s} = -(\vec{s} + \mu g_K(\vec{x}))
\]

\[
A^T \Delta \vec{y} + \Delta \vec{s} = 0,
\]

where \(w\) is the scaling point of \((\vec{x}, \vec{s})\).
Let $\beta, \delta \in (0, \frac{1}{2})$ be fixed constants such that
\[
\frac{7(\beta^2 + \delta^2)}{1 - \beta} \leq \left(1 - \frac{\delta}{\sqrt{\nu K}}\right) \beta, \quad \frac{2\sqrt{2}\beta}{1 - \beta} \leq 1.
\]

Algorithm PD(A)

(i) Let $(\tilde{x}, \tilde{y}, \tilde{s})$ be as in Proposition 4.6
(ii) If $-A^T y \in \text{int}(C)$ then HALT and return $y$ as a feasible solution for $A^T y \in \text{int}(C)$.
(iv) If $\sigma_{\min}(AH_C(x)^{-1}A^T) > \tau$ then put $\tilde{x} := x + H_C(x)^{-1}A^T(\nu_K)^{-1}x''$, HALT and return $\tilde{x}$ as a feasible solution for $Ax = 0$, $x \in \text{int}(C)$.
(vi) Update $(\tilde{x}, \tilde{y}, \tilde{s})$ as in (11), (12) for $\mu = \tilde{\mu}$.
(vii) Go to (ii).

Proof of Theorem 3.1. Arguments that are now standard in interior-point theory such as those in [18, Sec. 3.7] or [19, Sec. 3] ensure that the iterates generated by Algorithm PD lie in $N_{\beta}$ and that $\mu(\tilde{x}, \tilde{s})$ is reduced by $(1 - \frac{\delta}{\sqrt{\nu K}})$ at each iteration.

In addition, because $c^T \tilde{x} - \bar{b}^T \tilde{y} = \tau + \eta = \nu_K \mu(\tilde{x}, \tilde{s})$, Propositions 4.4 and 4.5 ensure that the algorithm halts as soon as $\mu(\tilde{x}, \tilde{s})$ falls below the threshold $(1 - \frac{\delta}{\sqrt{\nu K}^\rho(A)})$ (possibly sooner).

Since $\mu(\tilde{x}, \tilde{s})$ is reduced by $(1 - \frac{\delta}{\sqrt{\nu K}^\rho(A)})$ at each iteration, and at the initial point $\mu(\bar{x}, \bar{s}) = 2 \frac{||Ae||}{\beta} = \frac{||Ae||}{\beta \sqrt{\nu K}^\rho(A)}$, the threshold $(1 - \frac{\delta}{\sqrt{\nu K}^\rho(A)})$ is reached within
\[
O(\sqrt{\nu K} \log(\nu_K \sqrt{\nu K} ||Ae||/\rho(A))) = O(\sqrt{\nu K} \log(\nu_K C(A)))
\]
iterations. \[\square\]

4.5 On the equations arising at each interior-point iteration

The core of the computational work at each main iteration of Algorithm PD is the solution of (12). This system is typically solved via the Schur complement: First, solve the reduced system
\[
(AH_K(w)^{-1}A^T)\Delta \tilde{y} = -AH_K(w)^{-1}(\tilde{s} + \mu g_K(\tilde{x})), \quad (13)
\]
and then set
\[
\Delta \tilde{s} = -A\Delta \tilde{y}, \quad \Delta \tilde{x} = H_K(w)^{-1}(\tilde{s} + g_K(\tilde{x}) - \Delta \tilde{s}).
\]
Observe that the numerically critical step above is the solution of the reduced system \( (13) \). The condition number of this system grows in a controlled manner throughout the algorithm, and remains always bounded away by a factor of \( C(A) \), as the next theorem states.

**Theorem 4.7** Assume \((\vec{x}, \vec{y}, \vec{s}) \in N_\beta\). Let \( w \) be the scaling point of \( \vec{x}, \vec{s} \), and put \( \bar{w} := \sqrt{\mu(\vec{x}, \vec{s})} w \). Then

\[
\sigma_{\min}(A K^{-1} w^{-1} A^T) \geq \left( \frac{1 - \frac{9}{4} \beta}{2(\nu_K + \beta)} \min \left\{ \rho_P(A) + (1 - \beta) \mu(\vec{x}, \vec{s}), \frac{\sqrt{2}}{2} \right\} \right)^2
\]

and

\[
\sigma_{\max}(A K^{-1} w^{-1} A^T) \leq \left( \frac{\|A\|(3 + 2 \nu_K \mu(\vec{x}, \vec{y}))}{(1 - \frac{5}{4} \beta)} \right)^2
\]

In particular, if \( A \) is normalized, i.e., if \( \|A\| = 1 \), then

\[
\kappa(A K^{-1} w^{-1} A^T) = \kappa(A K^{-1} w^{-1} A^T) = O \left( \frac{\nu_K^2 \|A\|}{\mu(\vec{x}, \vec{s}) + \rho_P(A)} \right)^2,
\]

and hence \( \kappa(A K^{-1} w^{-1} A^T) \) is bounded above by a constant factor of \( \nu_K^4 C(A)^2 \) throughout Algorithm PD.

**Proof.** See Section 5.1.

Theorem 4.7 highlights one of the key features of our formulation. Only with a result of such kind it would be possible to derive results concerning numerical stability or effects of finite-precision arithmetic when solving the feasibility problems \([1], [2]\). Indeed, in [2] a variation of our approach provides the basis for an algorithm that solves \([1], [2]\) with a finite-precision machine in the special case \( C = \mathbb{R}_+^n \). The approach of this paper also serves as a basis for a similar result in the more general case when \( C \) is a direct product of second-order cones [3].

## 5 A perturbation theorem and proofs of main results

Throughout this section we use the following convenient notation: Given \( z \in \text{int}(K) \), we let \( B_z := \{ v : \|v\|_z \leq 1 \} \). In addition, to ease notation we shall write \( \mu \) as shorthand for \( \mu \).

### 5.1 A perturbation result for self-scaled programs

Theorem 5.1 provides the core of the proofs of Theorem 4.7 and Proposition 4.4. We note that Theorem 5.1 actually holds for general primal-dual pair of self-scaled programs, i.e., it holds for general \( A, \vec{b}, \vec{c} \).
Theorem 5.1 Assume $(\vec{x}, \vec{y}, \vec{s}) \in N_\beta$. Suppose $\bar{b} \not\in AB_{-\mu g K}(\vec{x})$. If $\alpha > \nu K + \beta$ then for either $\Delta b = \alpha \bar{b}$ or $\Delta b = -\alpha \bar{b}$ the optimal value of the perturbed problem

$$\min \; \langle \vec{c}, z \rangle$$

$$\mathcal{A} z = \bar{b} + \Delta b \quad z \in K$$

is greater than $\langle \vec{c}, \vec{x} \rangle$.

Proof. Choose either $\Delta b = \alpha \bar{b}$ or $\Delta b = -\alpha \bar{b}$ so that $\langle \vec{y}, \Delta b \rangle \geq 0$. Proceed by contradiction: suppose there exists $z$ such that

$$\langle c, z \rangle \leq \langle c, \vec{x} \rangle, \quad \mathcal{A} z = \bar{b} + \Delta b, \quad z \in K.$$

Then

$$\langle c, z - \vec{x} \rangle = \langle \mathcal{A}^T \vec{y} + \vec{s}, z - \vec{x} \rangle$$

$$= \langle \vec{y}, \mathcal{A}(z - \vec{x}) \rangle + \langle \vec{s}, z - \vec{x} \rangle$$

$$= \langle \vec{y}, \Delta b \rangle + \langle \vec{s}, z - \vec{x} \rangle. \quad (14)$$

But by Proposition 2.2(e), $\langle \vec{x}, \vec{s} \rangle = \mu \nu K = \langle -\mu g(\vec{s}), \vec{s} \rangle$. Consequently, (14) yields

$$\langle \vec{s}, z + \mu g(\vec{s}) \rangle = \langle \vec{s}, z - \vec{x} \rangle = \langle c, z - \vec{x} \rangle - \langle \vec{y}, \Delta b \rangle \leq 0.$$

Put $u := -\mu g(\vec{s})$. By Proposition 2.2(c), $g(u) = -\mu g(\vec{s})$ and hence

$$\langle g(u), z - u \rangle = \frac{1}{\mu} \langle \vec{s}, z + \mu g(\vec{s}) \rangle \geq 0.$$

Therefore, Proposition 2.2(b) yields

$$\nu K \geq \|z - u\|_u = \|z + \mu g K(\vec{s})\|_{-\mu g K}(\vec{s}).$$

On the other hand, $\|\vec{x} + \mu g K(\vec{s})\|_{-\mu g K}(\vec{s}) \leq \beta$ because $(\vec{x}, \vec{y}, \vec{s}) \in N_\beta$. Thus, by the triangle inequality,

$$\|z - \vec{x}\|_{-\mu g K}(\vec{s}) \leq \nu K + \beta.$$

Hence $\Delta b = \mathcal{A}(z - \vec{x}) \in (\nu K + \beta)AB_{-\mu g K}(\vec{s})$. Since $\Delta b = \alpha \bar{b}$ or $\Delta b = -\alpha \bar{b}$, and $|\alpha| > \nu K + \beta$, it follows that $\bar{b} \in AB_{-\mu g K}(\vec{s})$. This contradicts the hypothesis $\bar{b} \not\in AB_{-\mu g K}(\vec{s})$.

The proofs of Theorem 4.7 and Proposition 4.4 rely on the following observation: Given $z \in \text{int}(K)$

$$\sigma_{\min}(\mathcal{A}^T z^{-1} \mathcal{A}) = (\max\{\delta : \|u\| \leq \delta \Rightarrow u \in AB_z\})^2. \quad (15)$$

The proofs below also rely on the following two lemmas.

Lemma 5.2 Let $\tau > 0$ be given. Consider the system

$$\langle \vec{c}, z \rangle \leq \tau$$

$$\mathcal{A} z = \bar{b} + v \quad z \in K. \quad (16)$$
(a) If \( u \in \mathbb{R}^m \) is such that \( \|u\| \leq \tau + \rho_P(A) \) then for \( v = (u, 0, 0) \in \mathbb{R}^{m+n+1} \) \(16\) is feasible.

(b) If \( v \in \mathbb{R}^{m+n+1} \) is such that \( \|v\| \leq \frac{1}{2} \min\{\tau + \rho_P(A), \frac{\sqrt{2}}{2}\} \) then \(16\) is feasible.

Proof.

(a) This readily follows from Proposition 2.1 and the construction of \( \mathcal{A}, \vec{b}, \vec{c} \) and \( K \).

(b) Assume \( v = (u, r, w) \). By part (a) the system \(16\) has a solution \( z_1 \) for \( v = (2u, 0, 0) \). On the other hand, from the construction of \( \mathcal{A}, \vec{b}, \vec{c} \) it follows that \(16\) also has a solution \( z_2 = (0, 1 + 2r, -2w, 0, 0) \) for \( v = (0, 2r, 2w) \). Hence \(16\) has a solution \( \frac{1}{2}(z_1 + z_2) \) for \( v = (u, r, w) \).  

Lemma 5.3 Let \( (\vec{x}, \vec{y}, \vec{s}) \in \mathcal{N}_\beta \). Then
\[
\tau = (\vec{c}, \vec{x}) \geq (1 - \beta)\mu.
\]

Proof. Since \( (\vec{x}, \vec{y}, \vec{s}) \in \mathcal{N}_\beta \), in particular
\[
\| (\tau, x'') + \mu g_{m-1}(1, s'') \|_{-g_{m-1}(1, s'')} \leq \beta \mu. \tag{17}
\]
But
\[
H_{m-1}(-g_{m-1}(1, s'')) = \frac{1}{2} \begin{bmatrix}
1 + \|s''\|^2 & 2(s'')^T \\
2s'' & (1 - \|s''\|^2)I + 2ss^T
\end{bmatrix} = \frac{1}{2} Q^2,
\]
where \( Q \) is the matrix (cf. [11])
\[
Q = \begin{bmatrix}
\frac{1}{s''} & -\frac{1}{s''} \frac{(s'')^T}{1 - \|s''\|^2} + \frac{s''(s'')^T}{1 - \|s''\|^2} \frac{1}{s''} \\
\end{bmatrix}.
\]
Some straightforward calculations then yield
\[
\| (\tau, x'') + \mu g_{m-1}(1, s'') \|_{-g_{m-1}(1, s'')} = \frac{1}{\sqrt{2}} \left\| Q \begin{bmatrix}
\tau \\
x''
\end{bmatrix} - 2\mu e \right\|,
\]
Hence \(17\) implies that
\[
|\tau + (x'')^Ts'' - 2\mu| \leq \sqrt{2}\beta \mu < 2\beta \mu.
\]
Consequently,
\[
2(1 - \beta)\mu \leq \tau + (x'')^Ts'' \leq \tau + \|x''\| \|s''\| \leq \tau(1 + \|s''\|) \leq 2\tau.
\]
5.2 Proof of Theorem 4.7

From Proposition 2.2(a,f) it follows that \( B - \mu g_k(\vec{s}) \subseteq \frac{1}{(1-\beta)(1-\frac{5}{4}\beta)} B \bar{w} \subseteq \frac{1}{1-\frac{5}{4} \beta} B \bar{w} \).

Therefore, by [15], to prove the lower bound on \( \sigma_{\min}(AH(\bar{w})^{-1}A^T) \) it suffices to prove

\[
\|u\| > \frac{1}{2(\nu K + \beta)} \min \left\{ \rho_P(A) + (1 - \beta) \mu, \frac{\sqrt{2}}{2} \right\}.
\]

(18)

We next show (18). Suppose \( u \notin A B - \mu g_k(\vec{s}) \). Then by Theorem 5.1 it follows that for some \( v \) such that \( \|v\| = (\nu K + \beta)\|u\| \), the optimal value of the perturbed problem

\[
\min \langle \vec{c}, z \rangle \quad A z = \vec{b} + v \quad z \in K
\]

is greater than \( \langle \vec{c}, \vec{x} \rangle \). Thus Lemma 5.2(b) implies that

\[
\|v\| = (\nu K + \beta)\|u\| > \frac{1}{2} \min \left\{ \rho_P(A) + \langle \vec{c}, \vec{x} \rangle, \frac{\sqrt{2}}{2} \right\}.
\]

To finish, just observe that by Lemma 5.3 the latter expression is at least as large as \( \frac{1}{2} \min \left\{ \rho_P(A) + (1 - \beta) \mu, \frac{\sqrt{2}}{2} \right\} \).

For the upper bound on \( \sigma_{\max}(AH(\bar{w})^{-1}A^T) \), notice that, again by Proposition 2.2(a,b,c,f),

\[
\|H(\bar{w})^{-1}\| \leq \frac{\|H(\vec{x})^{-1}\|}{(1 - \frac{5}{4} \beta)^2} \leq \frac{\|\vec{x}\|^2}{(1 - \frac{5}{4} \beta)^2}.
\]

Thus,

\[
\sigma_{\max}(AH(\bar{w})^{-1}A^T) = \|AH(\bar{w})^{-1}A^T\| \leq \frac{\|A\|\|\vec{x}\|^2}{(1 - \frac{5}{4} \beta)^2} \leq \left( \frac{\|A\|(3 + 2\tau)}{1 - \frac{5}{4} \beta} \right)^2.
\]

5.3 Proof of Proposition 4.4

Let \( A_1 := [A \ 0 \ 0 \ 0 \ I_m] \). Observe that

\[
A_1 H_K(\vec{x})^{-1}A_1^T = AH_C(\vec{x})^{-1}A^T + [I_m \ 0] H_L(\vec{x}'', \tau)^{-1} [I_m \ 0].
\]
Thus
\[
\sigma_{\min}(AH_C(x)^{-1}A^T) \geq \sigma_{\min}(A_1H_K(\bar{x})^{-1}A_1^T) - \left\| \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \right\| H_{\text{im}}(x'',\tau)^{-1} \left\| \begin{bmatrix} I_m \\ 0 \end{bmatrix} \right\|.
\] (19)

But
\[
\left\| \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \right\| H_{\text{im}}(x'',\tau)^{-1} \left\| \begin{bmatrix} I_m \\ 0 \end{bmatrix} \right\| \leq \frac{\tau^2 + \|x''\|^2}{2} \leq \tau^2.
\] (20)

Therefore, to obtain the first part of Proposition 4.4 it suffices to prove the following inequality:
\[
\sigma_{\min}(A_1H(\bar{x})^{-1}A_1^T) \geq \left( 1 - \beta \right) (\rho_P(A) + \tau) \frac{\nu_K + \beta}{2}. \] (21)

Because from (19), (20), and (21) it follows that if \( \tau < \left( \frac{1 - \beta}{\sqrt{2(\nu_K + \beta)}} \right) \), then
\[
\sigma_{\min}(AH_C(x)^{-1}A^T) > \left( 1 - \beta \right) (\rho_P(A) + \tau)^2 - \tau^2 > \left( \frac{(1 - \beta)^2 \rho_P(A)}{2(\nu_K + \beta)^2} \right). \]

We next prove (21): By Lemma 5.2(a), for any given \( v \in \mathbb{R}^m \) such that \( \|v\| \leq \rho_P(A) + (\bar{c}, \bar{x}) \) the following system has a solution
\[
\langle \bar{c}, z \rangle \leq \langle \bar{c}, \bar{x} \rangle = \tau \\
A\bar{z} = \bar{b} + \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \\
z \in K.
\]

Thus, Theorem 5.1 implies that
\[
u \in \mathbb{R}^m, \|u\| \leq \frac{\rho_P(A) + \tau}{\nu_K + \beta} \Rightarrow \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \in A_{B_{-\mu g(\bar{s})}}.
\]

In other words,
\[
u \in \mathbb{R}^m, \|u\| \leq \frac{\rho_P(A) + \tau}{\nu_K + \beta} \Rightarrow u \in A_1B_{-\mu g(\bar{s})} \subseteq \frac{1}{1 - \beta} A_1B_{\bar{x}}.
\]

The last inclusion follows from Proposition 2.2(a) because \( \|x + \mu g(\bar{s})\|_{-\mu g(\bar{s})} < \beta \). Hence (21) follows from (15).
For the second part, observe that for \( \bar{x} := x + H_C(x)^{-1}A^T(AH_C(x)^{-1}A^T)^{-1}x'' \)
\[
\|\bar{x} - x\|^2_x = (x'', (AH_C(x)^{-1}A^T)^{-1}x'') \\
\leq \frac{\|x''\|^2}{\sigma_{\min}(AH_C(x)^{-1}A^T)} \\
< \frac{2(\nu_K + \beta)^2\tau^2}{(1 - \beta)^2\rho_P(A)^2} \\
< 1.
\]

Hence \( \bar{x} \in \text{int}(C) \) by Proposition 2.2(a). Thus \( \bar{x} \) satisfies
\[
A\bar{x} = Ax - x'' = 0, \bar{x} \in \text{int}(C), \|\bar{x} - x\|_x \leq \frac{\sqrt{2}(\nu_K + \beta)\tau}{(1 - \beta)\rho_P(A)}.
\]

\[\square\]

5.4 Proof of Proposition 4.5

Since \( (\bar{x}, \bar{y}, \bar{s}) \in \mathcal{N}_\beta \), the point \( \bar{x} = (x, t, x', \tau, x'') \) satisfies
\[
Ax + x'' = 0, x \in \text{int}(C)
\]
with \( \|x''\| \leq \bar{c}^T\bar{x} = \tau \). In particular, \( x \neq 0 \) and thus
\[
\left( A + \frac{x''x^T}{\|x\|^2} \right)x = 0, 0 \neq x \in C.
\]

Hence \( \left( A + \frac{x''x^T}{\|x\|^2} \right) \notin \mathcal{D} \), and consequently
\[
\rho_D(A) \leq \left\| \frac{x''x^T}{\|x\|^2} \right\| = \frac{\|x''\|}{\|x\|} \leq \frac{\tau}{\|x\|}.
\]

But since the optimal value of (10) is zero, \( \eta \geq 0 \) and consequently \( \tau \leq \tau + \eta = \langle \bar{c}, \bar{x} \rangle - \langle \bar{b}, \bar{y} \rangle = \langle \bar{x}, \bar{s} \rangle = \nu_K\mu \). Thus
\[
\|x\| \leq \frac{\tau}{\rho_D(A)} \leq \frac{\nu_K\mu}{\rho_D(A)}.
\]

Since \( (\bar{x}, \bar{y}, \bar{s}) \in \mathcal{N}_\beta \), we have \( \|\bar{s} + \mu g_K(\bar{x})\|_{-\mu g_K(\bar{x})} \leq \beta \). In particular, \( \|s + \mu g_C(x)\|_{-\mu g_C(x)} \leq \beta \). Thus, Proposition 2.2(a,c,d) yields
\[
\|H_C(s)\| \leq \frac{\|H_C(-\mu g_C(x))\|}{(1 - \beta)^2} = \frac{\|H_C(x)^{-1}\|}{(1 - \beta)^2\mu^2} \leq \frac{\|x\|^2}{(1 - \beta)^2\mu^2} \leq \frac{\nu_K^2}{(1 - \beta)^2\rho_D(A)^2}.
\]

On the other hand, again since \( (\bar{x}, \bar{y}, \bar{s}) \in \mathcal{N}_\beta \), the point \( \bar{y} = (y, -\eta, y') \) satisfies
\[
\| - A^Ty - s\|^2_s = \|y'\|^2_s \leq \eta^2\|H_C(s)\| \leq \eta^2\|H_C(s)\|.
\]

Therefore if \( \eta < \frac{(1 - \beta)\rho_D(A)}{\nu_K} \) we have \( \| - A^Ty - s\|^2_s < 1 \) and consequently \( -A^Ty \in \text{int}(C) \) by Proposition 2.2(a). \( \square \)
References


