Optimal Commodity Trading with a Capacitated Storage Asset

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Abstract

This paper considers the so called warehouse problem with both space and injection/withdrawal capacity limits. This is a foundational problem in the merchant management of assets for the storage of commodities, such as energy sources and natural resources. When the commodity spot price evolves according to an exogenous Markov process, this work shows that the optimal inventory trading policy of a risk neutral merchant is characterized by two stage and spot price dependent basestock targets. Under some assumptions, these targets are monotone in the spot price and partition the available inventory and spot price space in each stage into three regions, where it is respectively optimal to buy and inject, do nothing, and withdraw and sell. In some cases of practical importance, one can easily compute the optimal basestock targets. The structure of the optimal policy is nontrivial because in each stage the merchant’s qualification of high (selling) and low (buying) commodity prices in general depends on the merchant’s inventory availability. This is a consequence of the interplay between the capacity and space limits of the storage asset and brings to light the nontrivial nature of the interface between trading and operations. A computational analysis based on natural gas data shows that mismanaging this interface can yield significant value losses. Moreover, adapting the merchant’s optimal trading policy to the spot price stochastic evolution has substantial value. This value can be almost entirely generated by reacting to the unfolding of price uncertainty; that is, by sequentially reoptimizing a model that ignores this source of uncertainty.

1 Introduction

Many natural resources and energy sources are commodities. According to the Webster’s New Universal Unabridged Dictionary (1996 [45]), a commodity is “any unprocessed or partially processed good, as grains, fruits, and vegetables, or precious metals.” Coal, oil, and natural gas are additional examples. For a storable commodity, the economic interpretation of storage is the amount of commodity carried over to the next period from the current period; that is, storage from the prior period plus the difference between the commodity production and consumption in the current period (Williams and Wright 1991 [47]). Professional commodity storers, hereafter referred to as merchants, trade this surplus in wholesale markets that, for basic commodities, resemble a situation of perfect competition; that is, they are characterized by many small players who behave as price takers. For example, such a setting is the natural gas market at Henry Hub, Louisiana, the delivery location of the New York Mercantile Exchange (NYMEX) natural gas futures contract.

Merchants need access to storage facilities to support their commodity trading activities. They may own such facilities themselves, or hold contracts on their capacity. In this paper, a storage
asset refers to the facility where a commodity can be physically stored, or a contractual agreement that entitles its owner to usage of a portion of such a facility. These assets feature two distinctive characteristics. On the operational side, while the storage technology may take many forms (from conventional warehouses and oil tanks to underground depleted reservoirs, aquifers, and salt domes used to store natural gas), minimum/maximum inventory levels (space) and injection/withdrawal capacity limits are ubiquitous. On the financial side, commodity prices are notoriously variable and volatile (Seppi 2002 [35]), and storage assets give their managers (merchants) the real option (Trigeorgis 1996 [41]) to buy the commodity at one point in time, store it, and sell it at a later point in time to exploit price variability and volatility.

**Focus.** This paper focuses on the commercial management of a commodity storage asset. This requires determining an inventory trading policy that, given the current commodity spot price and inventory available in the storage facility, tells the merchant how much commodity to buy from the wholesale market and inject into this facility, or withdraw from this facility and sell into the wholesale market. This is a foundational problem in the commercial management of commodities, which has been studied in the literature on the warehouse problem. Cahn (1948 [5]) introduces this problem as follows: “Given a warehouse with fixed capacity and an initial stock of a certain product, which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales?” The fixed capacity attribute here refers to a finite size warehouse without restrictions on how fast it can be filled up or emptied. As will become apparent below, this qualification is critical. Charnes and Cooper (1955 [6]) discuss network flow formulations of this problem. Bellman (1956 [1]) presents a dynamic programming formulation, and Dreyfus (1957 [9]) shows that its optimal policy is simple: in each stage, given the current price, if it is optimal to trade then empty/fill-up the facility, and do nothing otherwise. This is a simple type of critical level, or, equivalently, basestock inventory trading policy. Charnes et al. (1966 [7]) extend this work to include stochastic prices and show that this property persists in this case.

**Motivation and research questions.** The warehouse problem literature ignores the important fact that storage facilities in practice often feature injection/withdrawal capacity constraints. That is, they are often capacitated in addition to having finite size. For example, this occurs in natural gas storage (see, e.g., Kobasa 1988 [22, pp. 8-9], Tek 1996 [39, Chapter 3], Maragos 2002 [26, p. 435], Geman 2005 [14, pp. 304-307]). This literature has also not yet studied how the optimal inventory trading decisions of a merchant depend on the commodity price.

Thus, the following research questions remain unanswered in the literature. What is the struc-
ture of the optimal inventory trading policy, both in terms of inventory availability and prevailing commodity price, when the storage asset features both space and capacity constraints? When trading is optimal, is it necessarily optimal to trade at capacity, as in the classical warehouse problem? In other words, is it always possible to decouple inventory trading and operational decisions? If this is not the case, what is the likely loss in value of ignoring the interface between trading and operations in practice? How important is the value of modeling the uncertainty in the commodity price evolution likely to be in practice?

Contributions. This paper contributes to the extant literature (reviewed below) by answering these questions. It presents and studies a finite horizon periodic review Markov decision process (MDP, Puterman 1994 [29], Heyman and Sobel 2004 [17], and Bertsekas 2005 [2]) whose states in each stage include both the merchant available inventory and the current commodity spot price, with the latter evolving as an exogenous Markov process. This MDP optimizes the inventory trading policy of a price taking merchant subject to space and capacity limits, which give rise to kinked injection/withdrawal capacity functions of inventory.

The analysis of this MDP significantly extends the work of Charnes et al. (1966 [7]) and establishes the optimality of a modified critical level (basestock target) policy. This policy is characterized by two stage and spot price dependent basestock targets that, under some assumptions, decrease in the spot price and partition the available inventory and spot price space in each stage into three regions, where it is respectively optimal to buy and inject (BI), do nothing (DN), and withdraw and sell (WS). In some cases of practical relevance, these targets can be easily computed.

This structure is nontrivial because, differently from Charnes et al. (1966 [7]), at the same commodity price and stage the type of the merchant’s optimal action can depend on its inventory availability. That is, at the same commodity price and stage the BI, DN, and WS actions can be optimal at different inventory levels. This nontriviality is a direct consequence of the kinked capacity functions. In fact, it disappears when the storage asset is uncapacitated as in Charnes et al. (1966 [7]), where optimal capacity underutilization can only occur at a trivial level; that is, when it is optimal to do nothing. Instead, with kinked capacity functions trading at capacity is generally suboptimal. Moreover, capacity underutilization can optimally occur at every inventory level for which trading (BI or WS) is optimal.

The nature of the BI/DN/WS structure implies that the interface between inventory trading and operational decisions is in general nontrivial; that is, these choices cannot be optimally decoupled by letting the inventory trader decide at what price to buy/sell and the operations manager inject/withdraw at capacity. This paper uses natural gas data to quantify the likely loss yielded by
decoupling these choices in practice. Here, it shows that while the addition of constraining injection/withdrawal capacity, and hence kinked capacity functions, does not dramatically reduce the value of a storage asset, mismanaging the trading operations interface can yield substantial value losses. Moreover, due to incorrect inventory trading decisions, increasing the injection capacity can magnify these losses.

Using this data, this paper also quantifies the value of modeling the uncertainty in the commodity price evolution. A model that ignores the stochastic evolution in the spot price can achieve about 79% of the value of the optimal policy obtained by considering the uncertain spot price evolution; that is, this model optimizes the inventory trading policy only once based on the initial constellation of futures prices (forward curve). This shows that there is significant value in adapting the optimal trading policy to the stochastic price evolution. Moreover, this value can be almost entirely generated by reacting to the unfolding of price uncertainty; that is, by reoptimizing at each decision time a model that ignores this uncertainty and only employs the current forward curve.

Relevance. It is insightful to interpret the nontriviality of the BI/DN/WS structure in terms of high and low commodity prices. The key insight here is that at a given decision time a merchant cannot always tell whether a given commodity price is high or low independently of its own inventory availability. In other words, at this time the same price can be both high and low at different inventory levels, an insight that is markedly different from the inventory independent price qualification implied by the analysis of Charnes et al. (1966 [7]). This occurs even if the merchant is risk neutral and price taker. Thus, nonlinearities in the operations of a commodity storage asset can fundamentally condition a merchant qualification of high and low commodity prices. In this paper, the relevant nonlinearities in the storage asset operations are brought about by the interplay between the space and capacity limits that give rise to kinked capacity functions.

The finding that mismanaging the interface between trading and operations is likely to yield substantial value losses is also relevant. The surprising part of this result is that it occurs even when the addition of constraining injection/withdrawal capacity does not dramatically reduce the value of a storage asset, which is due to the fact that in these cases the asset is operated at similar flow rates. Thus, nonlinear capacity functions make managing a commodity storage asset a genuinely difficult problem.

The quantification of the likely effect of price uncertainty on the value of a commodity storage asset points to the importance of exploiting changing price conditions in determining an inventory trading policy. The practical insight here is that this value can be captured in a fairly easy manner by simple sequential reoptimizations of a model that in fact ignores this uncertainty.
However, this result does not imply that modeling price uncertainty is not important in practice. For example, in North America the companies that own natural gas storage facilities rent them to merchants by entering into storage contracts with them. The valuation of these contracts is an important problem in practice. This requires computing the value of the possible cash flows during the contract lifetime. Ignoring price uncertainty in this valuation can significantly undervalue a contract. In other words, factoring this uncertainty in this valuation does require the explicit modeling of the commodity price stochastic evolution. What is less important is whether the optimization model used to generate these cash flows incorporates this uncertainty directly or indirectly. These insights should be valuable to merchants engaged in the management of natural gas storage assets.

Moreover, the results and insights of this paper have relevance beyond the natural gas storage industry. They remain pertinent for the management of storage assets for other (partially) storable commodities, such as oil, metals, and agricultural products (see, e.g., Geman 2005 [14]). While the facilities used to store these commodities may feature less stringent engineering restrictions in terms of injection/withdrawal capacities than natural gas storage facilities, in such cases capacity limitations equivalent to those studied in this paper can arise as a consequence of logistical and market constraints that limit how fast a merchant can effectively fill up or empty a facility.

**Novelty.** The novelty of this paper relative to the problem studied by Charnes et al. (1966 [7]) and their structural results has already been discussed above. In terms of methodological line of analysis, these authors use the concept of inventory evaluator, defined as the implicit marginal price of inventory carried into the future, and show that this quantity is constant in each stage due to the linearity in inventory of the optimal value function in their uncapacitated setting. Although this paper also uses a marginal analysis to establish the structure of the optimal policy, its analysis differs from theirs because the inventory evaluator is no longer constant in each stage when the storage asset is capacitated, which is due to the nonlinearity in inventory of the optimal value function in this case. Moreover, the analysis related to the price monotonicity of the optimal basestock targets and their computation in some specific cases appears to be novel.

Some of the policy structure findings of this paper are related to the basestock results available in the inventory management literature (Zipkin 2000 [49], Porteus 2002 [28]), in particular those works dealing with limited capacity (Federgruen and Zipkin 1986 [12], Kapuscinski and Tayur 1998 [20]), world-driven demand (Song and Zipkin 1993 [38]), and fluctuating purchasing costs (Scheller-Wolf and Tayur 1998 [32], Gavirneni 2004 [13]; see also the recent review by Haksoz and Seshadri 2007 [16]). The main focus of most of this literature, including Yang (2004 [48]) and Goel
and Gutierrez (2006 [15]) who deal with commodities in an *uncapacitated* setting, is on managing inventory to provide adequate service levels to customers in the face of demand uncertainty. In contrast, this paper deals with a different problem, that of optimizing the inventory trading decisions of merchants that operate in wholesale commodity markets to take advantage of price fluctuations. Thus, the model of this paper does not feature an explicit demand component and its related underage and overage costs, which fundamentally drive the basestock results available in the inventory management literature. Different from this literature, this work elucidates the role played in this respect by the interplay between the finite space and injection/withdrawal capacity of a storage facility.

While this study does not deal with capacity investment decisions, this paper is also related to the capacity management (investment) literature in terms of the form of the optimal policy structure (Van Mieghem 2003 [42]). However, aside from the obvious difference of problem domain, whereas the capacity management literature essentially places no constraints on the capacity adjustment choices, the constraints placed on the inventory adjustment decisions (the kinked capacity functions) play an important role in the analysis of this paper.

In the energy trading literature, Weston (2002 [46]), de Jong and Walet (2004 [8]), and Manoliu (2004 [25]) deal with the problem of optimally managing natural gas storage assets. In contrast to these papers, this paper formally establishes the structure of the optimal inventory trading policy. In addition, these papers do not study the issues related to the value of interfacing trading and operational decisions and how one can capture the value of exploiting price uncertainty by sequentially reoptimizing a deterministic model.

This paper is also related to Wang et al. (2008 [44]) who deal with the valuation of liquefied natural gas storage assets. The difference between these papers is that the incoming shipments (“injections”) are exogenous random variables in Wang et al. (2008 [44]), whereas the injected amount is a decision variable in this paper.

**Organization.** Section 2 presents the MDP model. Section 3 studies the structure of the optimal inventory trading policy and its computation. Section 4 quantifies the value of interfacing trading and operational decisions, the value of exploiting price uncertainty, and how much of this value can be captured by sequentially reoptimizing a deterministic model in the context of natural gas storage. Section 5 concludes.
2 Model

This section describes a periodic review model where inventory trading decisions are made at given equally spaced points in time belonging to finite set $T$. Set $\mathcal{J} = \{1, \ldots, J\}$ indexes them; that is, the $j$-th decision, $j \in \mathcal{J}$, is made at time $\tau_j \in T$. The length of each review period is $T$. An action is denoted as $a$: a positive action corresponds to a purchase followed by an injection, a negative action to a withdrawal followed by a sale, and zero is the do nothing action. The operational action, that is, an injection or a withdrawal, corresponding to a decision made at time $\tau_j$ is executed during the time interval in between times $\tau_j$ and $\tau_{j+1}$. This means that the receipt/delivery leadtime is zero, so that commodity purchased/sold at time $\tau_j$ is available/unavailable in storage at time $\tau_{j+1}$. For the most part, a buy-and-inject or withdraw-and-sell action will be simply referred to as an injection or a withdrawal, or a trade. The monetary payoff of the $j$-th trade occurs at time $\tau_j$. This modeling choice reflects typical practice, not only in commodity markets, where financial payoffs are accounted for at specific points in time, even though physical operations occur over time. However, if $T$ is chosen sufficiently small, this modeling set up can closely approximate those cases where the financial payoff occurs simultaneously with the operational execution of the trade.

The storage asset features minimum and maximum inventory levels, $\underline{x}$ and $\overline{x} \in \mathbb{R}_+$, with $\underline{x} < \overline{x}$ ($\underline{x} > 0$ is common in energy applications). Hence, the feasible inventory set is $\mathcal{X} := [\underline{x}, \overline{x}]$. There are constant injection and withdrawal capacities $\mathcal{C} > 0$ and $\mathcal{-C}$, respectively, on the maximum amount of inventory that can be injected into and withdrawn out of the facility in each review period (to be strict this applies to the absolute value of $\mathcal{C}$). It is assumed that both $\mathcal{C}$ and $-\mathcal{C}$ belong to set $(0, \overline{x} - \underline{x}]$. This implies that the storage asset features inventory dependent injection and withdrawal capacity functions of inventory $\Pi(x) : \mathcal{X} \to [\mathcal{C} \wedge (\overline{x} - x)]$ and $\mathcal{A}(x) : \mathcal{X} \to [\mathcal{C} \vee (\underline{x} - x), 0]$ defined as

\[
\Pi(x) := \mathcal{C} \wedge (\overline{x} - x) \\
\mathcal{A}(x) := \mathcal{C} \vee (\underline{x} - x).
\]

They express the maximum amount of commodity that can be injected and withdrawn, respectively, into and out of the facility during each review period starting from inventory level $x \in \mathcal{X}$ while keeping the inventory level in set $\mathcal{X}$ (strictly speaking this comment applies to the absolute value of $\mathcal{A}(x)$). The maximal capacity case arises when $\mathcal{C} = -\mathcal{C} = \overline{x} - \underline{x}$. This is the case considered by Charnes et al (1966 [7]) in which the facility can be filled up or emptied starting from every feasible inventory level in a single review period. Thus, this case is also labeled the “fast” or
Figure 1: The capacity functions and the feasible inventory action set. The solid lines are the capacity functions in the slow facility case, the thick dashed lines extend them to include the fast facility case. The inventory action set in each case is the grey area enclosed by the capacity functions.

"uncapacitated" facility case, and the previous case is refereed to as the "slow" or capacitated facility case.

Figure 1 illustrates the capacity functions in these cases. In the slow facility case, the kinks in these functions (at $x = \pi - C$ and $x = x - C$ in the injection and withdrawal cases, respectively) play a fundamental role in the analysis of the structure of the optimal trading policy carried out in §3. Obviously, these kinks are not present in the fast facility case.

At any review time, the sets of feasible withdrawal and injection actions, respectively, with inventory level $x \in \mathcal{X}$ are $\mathcal{A}^{FW}(x) := [g(x), 0]$ and $\mathcal{A}^{FI}(x) := [0, \pi(x)]$, and the set of all feasible actions is $\mathcal{A}^{F}(x) := \mathcal{A}^{FW}(x) \cup \mathcal{A}^{FI}(x)$. Figure 1 also illustrates the feasible inventory action set $\mathcal{C} := \{(x, a) : x \in \mathcal{X}, a \in \mathcal{A}^{F}(x)\}$, both in the fast and slow cases.

Let the commodity spot price be random variable $\tilde{s}_j$, with $\mathcal{S}_j \subseteq \mathbb{R}_+$ the set of its possible realizations (for notational simplicity $\tau_j$ is abbreviated to $j$ when subscripted). The Markovian stochastic process $\{\tilde{s}_j, j \in \mathcal{J}\}$ models the evolution of the spot price, with $\tilde{s}_1$ a degenerate random variable, and is independent of the merchant trading decisions. This is consistent with the assumption of a price taking merchant; that is, each trade is small relative to the size of the commodity market.
Figure 2: The immediate payoff function; $s^1 \leq s^2$.

A trading decision $a$ at time $\tau_j$, $j \in \mathcal{J}$, depends on the realized spot price, $s \in \mathcal{S}_j$, and, by the restriction $a \in \mathcal{A}^F(x)$, the available inventory, $x$, at this time. The inventory trading and adjustment, or simply immediate, payoff of this decision is $p_j(a, s) : \mathbb{R} \times \mathcal{S}_j \rightarrow \mathbb{R}$. Let $\alpha^W \in (0, 1]$ and $\alpha^I \geq 1$ commodity price adjustment factors. Even though the merchant is price taker, these adjustment factors are useful to model in kind fuel costs that arise in the context of natural gas storage. Letting $c^W$ and $c^I$ be positive constant marginal withdrawal and injection costs, respectively, the immediate payoff function is

$$p_j(a, s) := \begin{cases} - (\alpha^W s - c^W) a & \text{if } a \in \mathbb{R}_- \\ 0 & \text{if } a = 0 \\ - (\alpha^I s + c^I) a & \text{if } a \in \mathbb{R}_+ \end{cases}, \quad \forall j \in \mathcal{J}, \ s \in \mathcal{S}_j.$$  

Figure 2 illustrates this function (while not shown in this figure, this function is not necessarily positive in a withdrawal action). This function is kinked in the action at zero. In contrast to the kinks in the capacity functions, this kink does not play an important role in the analysis of §3, in the sense that the structural characterization of the optimal policy would persist even without this kink; that is, if the immediate payoff function were linear in the action.

A unit cost $h$ for physically holding (but not financing) inventory is charged at each time $\tau_j$, $j \in \mathcal{J}$, against the inventory $x \in \mathcal{X}$ available at this time. Some of the analysis of this paper extends to the case of a convex holding cost function. Cash flows are discounted from time $\tau_j$ back to time $\tau_{j-1}$, $j \in \mathcal{J} \setminus \{1\}$, using deterministic discount factor $\delta_{j-1} \in (0, 1]$.

An optimal inventory trading policy for a risk neutral merchant can be obtained by solving a finite horizon MDP. Denote by $E_j[\cdot] := E[\cdot \mid \tilde{s}_j = s]$ time $\tau_j$ conditional expectation given spot price $s \in \mathcal{S}_j$, $\forall j \in \mathcal{J}$. Set $\mathcal{J}$ indexes the stages and the state space in stage $j$ is $\mathcal{X} \times \mathcal{S}_j$. Denote by $V_j(x, s)$ the optimal value function in stage $j$ and state $(x, s)$. The dynamic programming recursion
This formulation is interpreted as follows. In the last stage, only the withdraw-and-sell action is available. In the remaining stages, both the buy-and-inject and withdraw-and-sell actions are available, but at most one of these actions is performed. As pointed out in §3, this modeling choice is without loss of generality.

The assumption that the merchant is risk neutral remains relevant when the merchant is not risk neutral but the market that trades natural gas financial contracts, such as futures, is complete and arbitrage free. In this case the spot price should be interpreted to evolve according to the risk neutral measure and the discount factor should be interpreted as being computed using the risk free interest rate (see Duffie 2001 [10, Chapters 2 and 6]), so that model (1)-(3) computes the market value of the storage asset, that is, the real option to store, during the given finite time horizon. This is the approach taken in §4.

3 Analysis

This section analyzes the structure of the optimal inventory trading policy and discusses computational aspects. Before proceeding, it is important to point out that the immediate payoff function \( p_j(a, s) \) is superadditive with respect to two actions with opposite signs for each given spot price; that is, given \( a^W < 0 < a^I \) it holds that \( p_j(a^W, s) + p_j(a^I, s) \leq p_j(a^W + a^I, s) \), \( \forall j \in J, s \in S_j \). This property, which can be easily proven, implies that it is never optimal to perform two opposite trades in the same stage, so that MDP formulation (1)-(3) is without loss of generality.

It is useful to define the discount factors \( \delta^{i,j} := 1, \forall j \in J \setminus \{j\} \), and \( \delta^{i,k} := \delta^{i,k-1} \delta_{k-1} \), \( \forall j \in J \setminus \{J\}, k \in J \), with \( k > j \). The following natural assumption on the expected discounted spot price holds throughout. It ensures that the optimal value function is finite in every stage and state.

**Assumption 1** (Expected discounted spot price). It holds that \( \delta^{i,k} E[\tilde{s}_k|\tilde{s}_j = s_j] < \infty \), \( \forall j \in J \setminus \{J\}, k \in J \), with \( k > j \), and \( s_j \in S_j \).

In the ensuing analysis, properties of functions should be interpreted in the weak sense.
3.1 Optimal Basestock Targets

This subsection establishes the structure of the optimal inventory trading policy when the spot price in a given stage is kept constant. It is useful for the ensuing analysis to define the functions

\[
U_j(x, s) := 0, \quad \forall (x, s) \in \mathcal{X} \times \mathcal{S}_j
\]

\[
U_j(x, s) := \delta_j \mathbb{E}[V_{j+1}(x, \hat{s}_{j+1})|\hat{s}_j = s], \quad \forall j \in \mathcal{J} \setminus \{J\}, \quad (x, s) \in \mathcal{X} \times \mathcal{S}_j.
\]

Proposition 1 establishes a basic property of these functions and of the optimal value function that is used extensively below (its proof is in Online Appendix A.1).

**Proposition 1 (Concavity).** In every stage \( j \in \mathcal{J} \), the functions \( U_j(x, s) \) and \( V_j(x, s) \) are concave in \( x \in \mathcal{X} \) for each given \( s \in \mathcal{S}_j \).

Theorem 1 leverages Proposition 1 to establish the structure of the optimal policy.

**Theorem 1 (Optimal basestock targets).** In every stage \( j \in \mathcal{J} \), there exist critical inventory levels \( \underline{b}_j(s) \) and \( \bar{b}_j(s) \in \mathcal{X} \), with \( \underline{b}_j(s) \leq \bar{b}_j(s) \), which depend on price \( s \), such that an optimal action in each state \((x, s) \in \mathcal{X} \times \mathcal{S}_j\) is

\[
a^*_j(x, s) = \begin{cases} 
(\bar{b}_j(s) - x) \land C & \text{if } x \in [x, \bar{b}_j(s)] \\
0 & \text{if } x \in [\bar{b}_j(s), \underline{b}_j(s)] \\
(\underline{b}_j(s) - x) \lor C & \text{if } x \in (\underline{b}_j(s), \bar{b}_j(s)] 
\end{cases}.
\]

**Proof.** Consider any stage \( j \in \mathcal{J} \). Pick state \((x, s) \in \mathcal{X} \times \mathcal{S}_j\). In determining an optimal action in this state, relax the capacity constraints \( \underline{C} \leq a \leq \bar{C} \) on the decision variable \( a \). Following Porteus (2002 [28, p. 66]), let \( y = x + a \) be the decision variable. Since \( a = y - x \), the relevant “capacity unconstrained” optimization is

\[
\max_{y \in \mathcal{X}} p_j(y - x, s) - hx + U_j(y, s).
\]

Depending on whether \( y \geq x \) or \( y \leq x \), the respective objective function of this optimization is

\[
U_j(y, s) - (\alpha I_s + c^I)y + (\alpha I_s + c^I - h)x
\]

\[
U_j(y, s) - (\alpha W_s - c^W)y + (\alpha W_s - c^W - h)x.
\]

Thus, maximization (7) can be approached by finding optimal solutions to the problems

\[
\max_{y \in [x, \bar{b}_j(s)]} U_j(y, s) - (\alpha I_s + c^I)y + (\alpha I_s + c^I - h)x
\]

\[
\max_{y \in [\underline{b}_j(s), \bar{b}_j(s)]} U_j(y, s) - (\alpha W_s - c^W)y + (\alpha W_s - c^W - h)x,
\]
and taking the optimal solution to (7) to be the one with the highest objective function value.

In particular, at \( x = \underline{x} \) and at \( x = \overline{x} \) maximization (7) is respectively equivalent to

\[
\max_{y \in X} U_j(y, s) - (\alpha^I s + c^I)y + (\alpha^I s + c^I - h)x
\]

\[
\max_{y \in X} U_j(y, s) - (\alpha^W s - c^W)y + (\alpha^W s - c^W - h)x,
\]

which can be simplified to

\[
\max_{y \in X} U_j(y, s) - (\alpha^I s + c^I)y
\]

\[
\max_{y \in X} U_j(y, s) - (\alpha^W s - c^W)y.
\]

Let \( \overline{b}_j(s) \) and \( \underline{b}_j(s) \) be optimal solutions to (12) and (13), respectively. To characterize these optimal solutions, consider the function \( U_j(\cdot, s) \). By Proposition 1, the left derivative of \( U_j(x, s) \) with respect to \( x \) exists everywhere on the interior of set \( X \), denoted by \( X^o \), for each \( s \in S_j \). Denote this left derivative by \( U'_j(x, s) \):

\[
U'_j(x, s) := \lim_{\epsilon \downarrow 0} \frac{U_j(x, s) - U_j(x - \epsilon, s)}{\epsilon}, \ \forall (x, s) \in X^o \times S_j.
\]

This definition is extended to inventory levels \( \underline{x} \) and \( \overline{x} \) as follows:

\[
U'_j(\underline{x}, s) := \lim_{x \downarrow \underline{x}} U'_j(x, s), \ \forall s \in S_j
\]

\[
U'_j(\overline{x}, s) := \lim_{x \uparrow \overline{x}} U'_j(x, s), \ \forall s \in S_j.
\]

Thus, with these conventions, \( U'_j(x, s) \) is defined for all \( x \in X \) for each \( s \in S_j \).

The objective functions of maximizations (12)-(13) are clearly concave in \( y \) for each given \( s \). By Proposition 1, the function \( U'_j(y, s) \) decreases in \( y \) for each given \( s \). Moreover, it holds that \( \alpha^I s + c^I \geq \alpha^W s - c^W \). Thus, an optimal solution \( \overline{b}_j(s) \) to (12) is never greater than an optimal solution \( \underline{b}_j(s) \) to (13).

Consider maximization (7) for any \( x \in [\underline{x}, \overline{b}_j(s)] \). It is clear that \( \overline{b}_j(s) \) is an optimal solution to (10). It also holds that \( x \) is an optimal solution to (11) because \( \overline{b}_j(s) \) maximizes (9) when \( y \) can take any value in set \( X \), but every feasible solution to (11) must satisfy \( y \leq x < \overline{b}_j(s) \leq \underline{b}_j(s) \). Moreover, \( x \) is a feasible solution to (10), so that

\[
U_j(\overline{b}_j(s), s) - (\alpha^I s + c^I)\overline{b}_j(s) + (\alpha^I s + c^I - h)x \geq U_j(x, s) - hx.
\]

Since the optimal objective function value of (11) is \( U_j(x, s) - hx \), \( \overline{b}_j(s) \) optimizes (7), and \( a^*_j(x, s) = (\overline{b}_j(s) - x) \wedge \overline{c} \).
Consider maximization (7) for any $x \in [b_j(s), \bar{b}_j(s)]$. Inventory level $x$ optimizes both (10) and (11) because $\bar{b}_j(s)$ and $\bar{b}_j(s)$, respectively, maximize (8) and (9) on set $\mathcal{X}$ and $b_j(s) \leq x \leq \bar{b}_j(s)$. It follows that $a^*_j(x, s) = 0$.

The case when $x \in (\bar{b}_j(s), \bar{x}]$ can be dealt with analogously to the case when $x \in [x, b_j(s)]$. □

Theorem 1 brings to light the BI/DN/WS structure of the optimal policy illustrated in Figure 3: in every stage $j \in \mathcal{J}$ and for each given spot price $s \in \mathcal{S}_j$, the optimal basestock targets $\bar{b}_j(s)$ and $\bar{b}_j(s)$ partition the feasible inventory set into the three sets $[x, b_j(s))$, $[b_j(s), \bar{b}_j(s)]$, and $(\bar{b}_j(s), \bar{x}]$, where it is respectively optimal to buy and inject to reach an inventory level that is as close as possible to $\bar{b}_j(s)$, do nothing, and withdraw and sell to reach an inventory level that is as close as possible to $\bar{b}_j(s)$.

It is worthwhile to emphasize that with a fast facility the work of Charnes et al. (1966 [7]) implies that if one of the BI/DN/WS actions is optimal at some inventory level in a given stage and for a given spot price, then this action is also so at all other inventory levels. Thus, if BI is optimal the optimal basestock targets are both equal to $\bar{x}$, if WS is optimal they are both equal to $\underline{x}$, and if DN is optimal one is equal to $\underline{x}$ and the other one to $\bar{x}$. In contrast, with a slow asset the optimal basestock targets in a given stage and for a given spot price are not necessarily equal to the minimum or maximum inventory levels. This difference in the nature of the optimal basestock targets is due to the optimal value function being respectively linear and nonlinear in inventory, for each given spot price in every stage, in the fast and slow asset cases: in the former case the inventory evaluator of Charnes et al. (1966 [7]), which corresponds to the function $U'_j(\cdot, s)$ defined by (14)-(16), is constant in inventory; in the latter case it is not necessarily so.

These observations yield the following insights. In the slow facility case, at the same spot price any type of action can be optimal in a given stage depending on the available inventory. Thus, in general, it is impossible to provide an absolute characterization of a spot price in a given stage as low or high; that is, one at which BI or WS is optimal, respectively. Any such statement must be made relative to inventory availability. Instead, in the fast facility case in a given stage it is
indeed possible to define a price as low, intermediate (that is, DN is optimal at this price), and high independently of the available inventory.

In other words, in the fast facility case when trading (BI or WS) is optimal, it is optimal to fully utilize the capacity functions, in the sense that an optimal BI action is equal to \( \pi(x) \) and an optimal WS action is equal to \( g(x) \). In contrast, in the slow facility case, it is clear that when trading (BI or WS) is optimal it is in general optimal to underutilize the available capacity, as expressed by the capacity functions, at some inventory level (of course, optimal capacity underutilization at all inventory levels also occurs in the fast facility case when doing nothing is optimal). What may be less clear is that this capacity underutilization can occur at every inventory level for which trading is optimal. Example 1 illustrates this possibility.

**Example 1** (Slow injection capacity). In this example there are three time periods \( J = 3 \). To emphasize the role played by the capacity functions, let the injection/withdrawal marginal costs and the holding cost be zero, and the price adjustment factors and the discount factor be one. Thus, it holds that \( p_j(a, s) = -sa, j = 1, 2, 3 \). At each time \( \tau_j \), the spot price takes one of the values in set \( \{s^L, s^M, s^H\} \), with \( 0 < s^L < s^M < s^H \), where superscripts L, M, and H stand for “low,” “medium,” and “high,” respectively. The spot price dynamics are “medium”, “low”, and “high”; that is, \( S_1 = \{s^M\} \), \( S_2 = \{s^L\} \), and \( S_3 = \{s^H\} \), as illustrated in Figure 4.

Clearly, one would like to fill up the facility in stage two, at the low price, and sell the entire inventory (down to \( x \)) in stage three, at the high price. Suppose that it is possible to empty the facility (reach inventory level \( x \)) in a single time period; that is, the asset has a fast withdrawal capacity function. However, filling it up requires more than one period, but less than two periods;
that is, the asset has a slow injection capacity function. Thus, with an empty facility, in stage one it is optimal to buy inventory and partially fill up the facility, hence optimally underutilizing the injection capacity at each inventory level where a BI trade is optimal; with a full facility, in stage one it is optimal to partially withdraw and sell the available inventory, hence optimally underutilizing the withdrawal capacity. In both cases, in stage two it is optimal to buy additional inventory and completely fill up the facility. This basic intuition is now formalized.

Since when empty the facility can be filled up in less than two periods but in more than one period, it holds that $\bar{x} + 2\bar{C} > \bar{x}$ and $\bar{x} + \bar{C} < \bar{x}$. It is useful to rewrite these inequalities as

$$\bar{C} < \bar{x} - \bar{x} < 2\bar{C}. \quad (17)$$

In stage three, it holds that $V_3(x, s^H) = s^H(x - \bar{x})$ because $A^{FW}(x) = [\bar{x} - x, 0], \forall x \in X$. In
stage two, with \( x \in \mathcal{X} \) and \( a \in [\bar{x} - x, \bar{C} \land (\bar{x} - x)] \), it is easy to verify that \( v_2(x, a, s^L) = (s^H - s^L)a + s^H(x - \bar{x}) \). Since \( v_2(x, a, s^L) \) increases in \( a \), it follows that \( a^*_2(x, s^L) = \bar{C} \land (\bar{x} - x) \) and \( V_2(x, s^L) = (s^H - s^L)[\bar{C} \land (\bar{x} - x)] + s^H(x - \bar{x}), \forall x \in \mathcal{X} \). Finally, consider stage one and focus on inventory level \( \bar{x} \). For \( a \in [0, \bar{x} - \bar{C}] \), it holds that \( v_1(\bar{x}, a, s^M) = (s^H - s^M)a + (s^H - s^L)[\bar{C} \land (\bar{x} - x - a)] \). Since
\[
\bar{C} \land (\bar{x} - x - a) = \begin{cases} 
\bar{C} & \text{if } a \in [0, \bar{x} - \bar{C}) \\
\bar{x} - x - a & \text{if } a \in [\bar{x} - \bar{C}, \bar{x} - \bar{C}, \bar{x} - \bar{x}] 
\end{cases},
\]
it follows that
\[
v_1(\bar{x}, a, s^M) = \begin{cases} 
(s^H - s^M)a + (s^H - s^L)\bar{C} & \text{if } a \in [0, \bar{x} - \bar{C}) \\
(s^L - s^M)a + (s^H - s^L)(\bar{x} - x - a) & \text{if } a \in [\bar{x} - \bar{C}, \bar{x} - \bar{x}].
\end{cases}
\]
It holds that \( \bar{x} - \bar{x} - \bar{C} =: a^*_1(\bar{x}, s^M) = \arg\max_{a \in [\bar{x} - \bar{C}, 0]} v_1(\bar{x}, a, s^M) \) and, by (17), \( 0 < a^*_1(\bar{x}, s^M) < \bar{C} = \bar{a}(\bar{x}) \). Thus, in stage one at inventory level \( \bar{x} \) it is optimal to buy and inject without fully utilizing the injection capacity. Moreover, it holds that \( \bar{a}(\bar{x}) < -\bar{C} =: a^*_1(\bar{x}, s^M) \in \arg\max_{a \in [\bar{x} - \bar{C}, 0]} v_1(\bar{x}, a, s^M). \)

Thus, in stage one at inventory level \( \bar{x} \) it is optimal to withdraw and sell without fully utilizing the withdrawal capacity. Since \( \bar{x} + a^*_1(\bar{x}, s^M) = \bar{x} - \bar{C} \) and \( \bar{x} - \bar{x} + a^*_1(\bar{x}, s^M) = \bar{x} - \bar{C} \), it holds that \( \bar{b}_1(s^M) = \bar{b}_1(s^M) = \bar{x} - \bar{C} \) and \( a^*_1(\bar{x}, s^M) = \bar{x} - \bar{C}, \forall x \in \mathcal{X}. \) Thus, in stage one it is optimal to buy and inject without fully utilizing the injection capacity at every inventory level in the interval \([\bar{x}, \bar{x} - \bar{C}]\), and it is optimal to withdraw and sell without fully utilizing the withdrawal capacity at every inventory level in the interval \((\bar{x} - \bar{C}, \bar{x}]\). Figure 5 displays the behavior of this optimal action function and relates it to those of the injection and withdrawal capacity functions.

### 3.2 Price Monotonicity of the Optimal Basestock Targets

The optimal basestock targets are said to be monotonic in the spot price if they decrease in the spot price in each stage. Example 2 shows that an optimal basestock target does not always satisfy this property.

**Example 2** (Nonmonotonic optimal BI basestock target). Let \( J = 2, \bar{x} = 0, \) and \( \bar{F} = 1. \) Suppose that \( \alpha^I = \alpha^W = 1, c^I = 0.04, c^W = 0, \delta_1 = 1, h = 0, \) and \( \bar{C} = -\bar{C} = 1. \) This is a fast asset, so that \( V_2(x, s) = xs \) and \( U_1(x, s) = x\bar{s}_{2|1}(s) \), where, with a slight abuse of notation, \( \bar{s}_{2|1}(s) := E[\bar{s}_2|\bar{s}_1 = s] \). Assume that there are four possible spot prices in stage one, those in set \( S_1 = \{0.01, 2.9, 12\} \), and that \( \bar{s}_{2|1}(s) \) is equal to 0.0149, 2.1922, 9.0375, and 11.8498 for \( s = 0.01, 2, 9, \) and 12, respectively. In stage one, \( \bar{b}_1(s) \) and \( \bar{b}_1(s) \) are optimal solutions to \( \max_{y \in [0.1, 0.1]} \bar{s}_{2|1}(s) - (s + 0.04)y \) and \( \max_{y \in [0.1]} \bar{s}_{2|1}(s) - sy \), respectively. Thus, \( \bar{b}_1(s) \) is equal to 0 if the BI unit margin, \( \bar{s}_{2|1}(s) - (s + 0.04) \), is negative and to 1 if this quantity is positive; and \( \bar{b}_1(s) \) is equal to 0 if the
Table 1: BI and WS unit margins, optimal basestock targets, and optimal type of action in stage one in Example 2.

<table>
<thead>
<tr>
<th>Function</th>
<th>0.01</th>
<th>2</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{s}_{21}(s) - (s + 0.04)$</td>
<td>0.0049</td>
<td>0.1522</td>
<td>–0.0025</td>
<td>–0.1902</td>
</tr>
<tr>
<td>$s_{21}(s) - s$</td>
<td>0.0375</td>
<td>0.0375</td>
<td>0.0375</td>
<td>0.0375</td>
</tr>
<tr>
<td>$b_1(s)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$b_1(s_1)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Optimal type of action: DN, BI, DN, WS.

WS unit margin, $s_{2|1}(s) - s$, is negative and to 1 if this quantity is positive. Table 1 displays the BI and WS unit margins and their associated optimal basestock targets and type of action for all possible spot prices in stage one. This table shows that $b_1(s)$ behaves nonmonotonic in $s$.

What makes the function $b_1(s)$ nonmonotonic in $s$ in Example 2 is the nonmonotonic behavior of the function $s_{2|1}(s) - s$ in $s$. In this example, had the latter function decreased in $s$, the function $b_1(s)$ would have been monotonic in $s$. This suggests that the price monotonicity of the optimal basestock targets can be established by imposing more structure on the spot price process. In particular, the following analysis proceeds by imposing on this process the conditions stated in Assumption 2.

**Assumption 2 (Spot price process).** For every $j \in J \setminus \{J\}$, (a) the distribution function of random variable $\tilde{s}_{j+1}$ conditional on the spot price $s$ in stage $j$ stochastically increases in $s \in S_j$; (b) the function $\delta_j E_j[\alpha^{\tilde{s}_{j+1}|\tilde{s}_j = s}] - \alpha^Ws$ decreases in $s \in S_j$.

The condition stated in part (a) of Assumption 2 is rather natural. In particular, it implies that the expected spot price in the next stage increases in the spot price in the current stage (see, e.g., Corollary 3.9.1(a) in Topkis 1998 [40]). Part (b) of Assumption 2 is motivated by the observation made after Example 2 and implies that the discounted expected spot price in the next stage increases at a slower rate than the spot price in the current stage (this is established in Lemma 3 in Online Appendix A.2). The well known mean reverting model of commodity prices (Seppi 2002 [35], Jaillet et al. 2004 [19]), which is used in §4, satisfies the condition stated in part (a) of Assumption 2 by Theorem 4 in Müller (2001 [27]), but not that stated in part (b). (Indeed, the expected spot prices in stage two conditional on the stage one spot prices used in Example 2 were obtained by using a mean reverting model.) A model that satisfies both of these conditions is geometric Brownian motion with appropriate drift rate. Schwartz and Smith (2000 [34]) use this model to represent long-term variations in commodity prices and suggest that it may be an
appropriate model for the valuation of long-term real options. However, the condition stated in part (b) of Assumption 2 is not necessary for the optimal basestock targets to be monotonic in price, as it is easy to verify that $b_1(s)$ would be monotonic in $s$ in Example 2 if $c_1$ were equal to zero. Theorem 2 establishes the price monotonicity of the optimal basestock targets when Assumption 2 holds.

**Theorem 2** (Price monotonicity). If Assumption 2 holds, then in every stage $j \in J$ the optimal basestock target functions $b_j(s)$ and $\bar{b}_j(s)$ decrease in the spot price $s \in S_j$.

The proof of this theorem is somewhat technical and is given in Online Appendix A.2. The main idea in this proof is to show that under Assumption 2 the functions $U_j'(y, s) - (\alpha I s + c_1)$ and $U_j'(y, s) - (\alpha W s - c_w)$ that are used to determine the optimal basestock targets in each stage $j$ (see (12)-(13)) decrease in the spot price $s$ (for each given $y$). This proof also shows that under Assumption 2 the function $U_j'(y, s)$ increases in the spot price $s$ in each stage $j$ (for each given $y$). This means that although the marginal expected value of inventory increases in the spot price, the marginal expected value of inventory net of its marginal acquisition cost and marginal disposal “cost” decreases in the spot price. Consequently, the optimal basestock targets decrease in the spot price, or, equivalently, the optimal amount of inventory bought and injected (respectively, withdrawn and sold) in each stage decreases (respectively, increases) in the spot price. Incidentally, this is consistent with the discussion of complementarity in Topkis (1998 [40, pp. 92-93]).

When Assumption 2 holds, Theorem 2 brings to light the structure of the optimal policy illustrated in Figure 6: in every stage $j \in J$, the optimal basestock targets $b_j(s)$ and $\bar{b}_j(s)$ decrease in the spot price $s \in S_j$ and partition the state space into disjoint BI, DN, and WS regions. Given the discussion following the proof of Theorem 1, how these functions decrease in the spot price can differ significantly in the slow and fast facility cases, as shown in panels (a) and (b) of Figure 6.

### 3.3 Computation

In applications, it is computationally useful to represent the evolution of the spot price as a Markov process that in each stage can only take on a finite number of values, for example using a lattice model such as in Jaillet et al. (2004 [19]; see also Luenberger 1998 [24, Chapter 12] and Hull 2000 [18, Chapter 16]). Thus, in the following, the spot price is assumed to evolve as stated in Assumption 3.

**Assumption 3** (Finite spot price sets). In every stage $j \in J$, the spot price set $S_j$ is finite.
This assumption implies that the random variable $\tilde{s}_{j+1}$ conditional on each spot price $s \in S_j$ in stage $j$ has a discrete probability distribution, $\forall j \in J \setminus \{J\}$. Proposition 2, whose proof is in Online Appendix A.3, establishes a useful property of the optimal value function under Assumption 3.

**Proposition 2 (Piecewise linearity).** Suppose that Assumption 3 holds. Then, in every stage $j \in J$, the optimal value function $V_j(x, s)$ is piecewise linear and continuous in $x \in X$ for each $s \in S_j$.

Under Assumption 3, computing the optimal basestock targets is facilitated by the capacities, $-C$ and $C$, and the inventory limits, $x$ and $\tau$, being integer multiples of some real number. In this case, Proposition 3, proved in Online Appendix A.3, establishes two useful properties of the optimal value function and basestock targets by leveraging the property established in Proposition 2.

**Proposition 3 (Restricted capacities and inventory limits).** Suppose that Assumption 3 holds and that there exists a maximal number $Q \in \mathbb{R}_+$ such that $-C$, $C$, $x$, and $\tau$ are all integer multiples of $Q$. Then, in every stage $j \in J$ and for each given spot price $s \in S_j$, (a) the optimal value function $V_j(x, s)$ can change slope in inventory $x$ only at inventory levels that are integer multiples of $Q$, and (b) the optimal basestock targets $\underline{b}_j(s)$ and $\overline{b}_j(s)$ can be taken to be integer multiples of $Q$. 

---

Figure 6: Illustration of the price monotonicity of the optimal basestock targets under Assumption 2; $\underline{s}$ and $\overline{s}$ are hypothetical minimum and maximum spot prices and the stage subscript is suppressed.
The properties established in Proposition 3 are useful because, under Assumption 3, in every stage one can compute the optimal basestock targets for each spot price by restricting attention to a finite number of feasible inventory levels, namely those that are multiples of the stated $Q$. Thus, one needs to compute the optimal value function in each stage and for each possible spot price only for the $1 + (\pi - \bar{\pi})/Q$ feasible inventory levels $\bar{\pi}, \bar{\pi} + Q, \bar{\pi} + 2Q, \ldots, \pi$. This can be easily done by optimally solving a discrete space MDP by standard backward recursion.

4 Natural Gas Storage Application

This section quantifies the managerial relevance of the BI/DN/WS structure through a computational analysis in the context of natural gas storage. Three issues are investigated: (1) the relevance of taking the capacity limits into account when determining a trading policy; that is, of optimally interfacing operational and inventory trading decisions; (2) the value of modeling price uncertainty; and (3) how much of this value can be captured by reacting to the unfolding of the stochastic spot price evolution through sequential reoptimizations of a deterministic model that only uses information about average price dynamics; that is, without computing an optimal policy.

**Spot price model.** The ensuing analysis assumes that the natural gas price evolves as a single factor mean reverting process with deterministic monthly seasonality factors as in Jaillet et al. (2004 [19]). The simpler model without the seasonality factors is a well known model of commodity price evolution discussed by Schwartz (1997 [33]), Smith and McCardle (1999 [37]), and Seppi (2002 [35]), among others. In particular, de Jong and Walet (2004 [8]) and Manoliu (2004 [25]) employ this model in the context of natural gas storage valuation.

In the model of Jaillet et al. (2004 [19]), the spot price at time $t$ is the exponential of a single mean reverting factor multiplied by a deterministic monthly seasonality factor. The single factor, whose value at time $t$ is denoted by $\chi_t$, is the natural logarithm of the spot price. It evolves in continuous time and space according to the stochastic differential equation

\[ d\chi_t = \kappa(\xi - \chi_t)dt + \sigma dZ_t, \tag{18} \]

where $\xi$, $\kappa > 0$, and $\sigma > 0$, respectively, are the long term mean reversion level, the speed of mean reversion, and the volatility of $\chi_t$, and $dZ_t$ is an increment to a standard Brownian motion. Notice that, as discussed by Schwartz (1997 [33]) and Jaillet et al. (2004 [19]), one can assume that the dynamics specified by (18) are under the risk neutral measure. The dynamics of the single factor under the objective measure would also be mean reverting but would feature a different mean reversion level (see also Ross 1997 [30], Schwartz 1997 [33], Seppi 2002 [35], and Smith 2005 [36]).
Table 2: Estimates of the seasonal mean reverting price model parameters based on NYMEX natural gas data from February 2006 (Source: Wang 2008 [43]).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long term log level ($\xi$)</td>
<td>1.9740</td>
</tr>
<tr>
<td>Speed of mean reversion ($\kappa$)</td>
<td>0.7200</td>
</tr>
<tr>
<td>Volatility ($\sigma$)</td>
<td>0.6610</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Monthly Seasonality Factors</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>1.1932</td>
</tr>
<tr>
<td>February</td>
<td>1.1948</td>
</tr>
<tr>
<td>March</td>
<td>1.1412</td>
</tr>
<tr>
<td>April</td>
<td>0.9142</td>
</tr>
<tr>
<td>May</td>
<td>0.8984</td>
</tr>
<tr>
<td>June</td>
<td>0.8998</td>
</tr>
<tr>
<td>July</td>
<td>0.9113</td>
</tr>
<tr>
<td>August</td>
<td>0.9219</td>
</tr>
<tr>
<td>September</td>
<td>0.9287</td>
</tr>
<tr>
<td>October</td>
<td>0.9412</td>
</tr>
<tr>
<td>November</td>
<td>1.0252</td>
</tr>
<tr>
<td>December</td>
<td>1.1049</td>
</tr>
</tbody>
</table>

The estimates of the parameters of this model employed in this paper are those obtained by Wang (2008 [43]) using NYMEX natural gas futures and option prices pertaining to February 2006. Table 2 displays these estimates.

**Storage asset parameters.** The minimum and maximum inventory levels $x$ and $\bar{x}$ are 0 and 10, respectively. (The unit of measurement of inventory is left unspecified on purpose and should be interpreted as an appropriate MMBTU multiple, where 1 MMBTU = 1,000,000 British thermal units.) In practice, the number of days needed to fill up and empty different types of natural gas storage facilities varies considerably, between 20-250 and 10-150, respectively (FERC 2004 [11, p. 7]). Moreover, the maximum injection and withdrawal quantities can be nonconstant functions of inventory for certain types of facilities (see, e.g., Maragos 2002 [26, p. 434]).

However, this is not always the case. For example, Levary and Dean (1980 [23], in the injection case), and Bopp et al. (1996 [3]), Knowles and Wirick (1998 [21]), Weston (2002 [46]), de Jong and Walet (2004 [8]), Manoliu (2004 [25]), and Byers (2006 [4]) use constant capacities, which is also how some contracts for the usage of natural gas storage facilities are written. This will also be the case in the ensuing analysis, where the monthly injection and withdrawal capacities are varied from 10% to 100% of the maximum available space (10 units) in increments of 10%. This setting covers several cases of practical interest since it corresponds to varying the number of days required to fill-up/empty a facility roughly between 30 and 300. It also gives rise to kinked capacity functions,
except when the facility is fast.

As explained by Maragos (2002 [26, p. 436]), in natural gas storage fuel is in kind since “every time gas is injected or withdrawn, the pumps of the facility use some of that gas as fuel.” Denote by $\varphi^I, \varphi^W \in [0, 1]$ injection and withdrawal loss factors, respectively. In order to have available in storage an amount $\hat{a}^I$ after an injection, one needs to inject an amount $\hat{a}^I$ that satisfies $\hat{a}^I = a^I + \varphi^I \hat{a}^I$; that is, $\hat{a}^I = a^I/(1 - \varphi^I)$. To deliver an amount $|\hat{a}^W|$ out of storage, one needs to withdraw a quantity $|a^W|$ such that $|a^W| = |\hat{a}^W| + \varphi^W |a^W|$; that is, $|\hat{a}^W| = |a^W|(1 - \varphi^W)$. In words, one needs to purchase more commodity than what will be available in storage after injecting it, and withdraw more commodity than what can be sold after withdrawing it. Hence, one can define the price adjustment factors $\alpha^W$ and $\alpha^I$ as $1 - \varphi^W$ and $1/(1 - \varphi^I)$, respectively. Fuel costs in natural gas storage are small. For example, Maragos (2002 [26, p. 436]) reports that “[t]ypically, the consumed fuel is no more than 1% of the injected or withdrawn gas.” Assuming that compression is used for injections but not for withdrawals, which is consistent with how some natural gas storage contracts are written in practice, the injection fuel loss factor is set equal to 0.01, and there is no loss associated with withdrawals, so that $\alpha^I = 1/0.99$ and $\alpha^W = 1$.

In practice, the inventory adjustment cost tends to be relatively small. For example, Maragos (2002 [26, p. 436]) reports that this cost “pays for the use of pumps and other storage facility equipment” and “usually runs no higher than one or two cents per mmBtu,” which is roughly about 0.10% of the futures prices displayed in Figure 7 (Table 3 in Online Appendix B reports the numerical values displayed in this figure). Thus, the withdrawal and injection marginal costs $c^W$ and $c^I$ are both set equal to $0.02/MMBtu$. Consistent with how natural gas storage assets are leased in the U.S. the holding cost is zero: $h = 0/MMBTU$.

**MDP.** The ensuing analysis features an MDP formulation with 24 stages ($J = 24$) corresponding to a monthly partition of a two year time period. The first stage corresponds to the beginning of March and the remaining ones to the beginning of each of the next 23 months. The evolution of the spot price during these months is represented by a trinomial lattice calibrated by standard methods (Jaillet et al. 2004 [19]) to the first 24 NYMEX natural gas futures prices observed at the end of 2/1/2006 illustrated in Figure 7. Since this trinomial lattice is built using the risk neutral measure, these initial futures prices can be thought of as the conditional expectation of the spot price in each remaining stage under this measure given the spot price in the first stage; this is taken to be $8.723/MMBtu$, the closing price for the March 2006 contract on 2/1/2006. The trinomial lattice models the stochastic variability around the expected spot price. The monthly discount factor is 0.9958, which corresponds to an annual risk free interest rate of 5% with continuous dis-
The value of optimally interfacing operational and trading decisions. The first issue to be investigated is the relevance of taking the capacity limits into account when determining a trading policy. In other words, it is of interest to measure the importance of optimally interfacing operational and inventory trading decisions. Three types of policies are considered for this purpose.

1. The fast capacity optimal policy (FCOP) that assumes that the injection/withdrawal capacity scale factors (I/WCSFs) are both always equal to 100%.

2. The slow capacity optimal policy (SCOP).

3. A decoupled operations and trading policy (DOTP) that uses the FCOP buying and selling prices in each stage, but whose actions are restricted by the capacity constraints. In other words, this policy buys and sells whenever FCOP does so, but its actions are constrained by the capacity functions. That is, it does trades at capacity, as given by the capacity functions, but it cannot always empty/fill-up the asset in a single stage. Thus, this policy mismanages the interface between operational and trading decisions in the merchant management of the storage asset: it models a situation where the trader computes buying/selling prices ignoring the asset capacity constraints, trading occurs at capacity according to the capacity functions, and the operations manager also injects/withdraws at capacity according to these functions. The value function of this policy can be easily computed.
Figure 8: Percent gains on the total value of the asset of FCOP relative to SCOP for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow.

The following discussion pertains to comparisons of the relevant value functions and other quantities of interest in stage one, when there is only one possible spot price, with zero initial inventory. The value function in this stage and state under a given policy is refereed to as the total asset value under this policy (this total value is the sum of the asset intrinsic and extrinsic values, which are defined later in this section). The ensuing discussion is based on Figures 8-14. (Tables 4-10 in Online Appendix B report the numerical values displayed in these figures.)

It is clear that the FCOP value function dominates that of SCOP in each stage. In fact, the SCOP value function increases when more capacity is available and becomes equal to that of FCOP when ICSF and WCSF are both equal to 1.0. Figure 8 displays the FCOP percentage gains on the total value of the asset relative to SCOP for each relevant ICSF and WCSF combination. As expected, these gains decrease when ICSF or WCSF increase. When ICSF and WCSF are at least 0.4 and 0.5, respectively, these gains are no more than 10%. These gains increase rapidly when WCSF or ICSF decrease below 0.3. To explain these observations, Figure 9 displays the ratio of the FCOP and SCOP flow rates. These flow rates are computed by folding backward the expected amounts of natural gas withdrawn during each stage by FCOP and SCOP. Focusing on the withdrawn natural gas is appropriate since these policies start with zero inventory and sell all the available inventory in the last stage. That is, one would obtain the same flow rates by focusing
on the amounts of injected natural gas.

The good performance of SCOP relative to FCOP when ICSF and WCSF are at least 0.4 and 0.5, respectively, is due to the fact that in these cases the two policies have very similar flow rates, with the FCOP flow rate, which is constant for each of the considered ICSF and WCSF combinations, being within 15% of the SCOP flow rates. It is interesting to note that in these cases the SCOP flow rates do not significantly increase by adding withdrawal capacity. Instead, the SCOP flow rates are significantly lower than the FCOP flow rate when WCSF or ICSF fall below 0.3, which explains the relatively poorer performance of SCOP in these cases.

One might be tempted to conclude that in most capacity configuration cases not much value would be lost by managing the operations and trading interface as in the DOTP; that is, trading as if this interface were not important. As now discussed, this conclusion is instead generally incorrect.

Figure 10 shows the percentage total asset value losses of DOTP relative to SCOP (that a loss can exceed 100% is due to the fact that the DOTP value function can be negative). This figure illustrates that decoupling operational and trading decisions can generate significant losses in the presence of capacity constraints. These losses are relatively contained, that is, below 10%, only when WCSF is equal to 1.0 and ICSF is 0.2 or larger, or when WCSF is 0.9 and ICSF is equal to 1.0. The deep losses displayed in Figure 10 result from purchases and injections made under the
Figure 10: Percent losses on the total value of the asset of DOTP relative to SCOP for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow when WCSF is equal to or greater than 0.5.

An incorrect assumption that any injected natural gas could be entirely withdrawn and sold at some later stage.

This is a basic mismatch between operational and trading decisions in the presence of capacity limits. As a consequence of this mismatch, more injection capacity is not necessarily beneficial for a given level of withdrawal capacity, when the latter capacity is “low.” For example, this can occur for WCSFs between 0.1 and 0.4. In other words, when the operations manager and the trader do not coordinate their decisions, it can be better to have a tighter injection capacity when the withdrawal capacity is “too low” because limited injection capacity avoids excessive inventory build up. Instead, a higher withdrawal capacity appears to be always beneficial, since it helps to dispose of previously accumulated inventory.

The value of price uncertainty. Figure 7 shows the pronounced deterministic variability (seasonality) in natural gas futures prices. Thus, one would expect that a large fraction of the value of managing a natural gas storage asset should derive from exploiting this variability; that is, predictable price spreads. The second issue to be investigated is the quantification of the value added by considering uncertain price variability to that yielded by taking advantage of seasonal price spreads. In the following comparisons, the relevant policy is SCOP.

Solving the MDP in each of the relevant cases using the deterministic price dynamics illustrated
Figure 11: Relative extrinsic values of the asset for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow when WCSF is equal to or greater than 0.4.

in Figure 7 allows one to measure the storage asset value attributable to seasonality. In financial engineering terminology, this is the intrinsic value of the storage asset; that is, the intrinsic value of the real option to store natural gas. Its corresponding policy amounts to deciding in the first stage the actions to be performed in each of the remaining stages. In practice, the financial trades associated with these actions can be implemented at the time the optimization takes place by forward buying/selling the relevant amounts of natural gas in the forward/futures market; that is, by implementing a “perfect hedge.”

The incremental value of stochastic price variability, also called extrinsic value in financial engineering terminology, is the difference between the total value of the asset and its intrinsic value. Figure 11 displays the extrinsic value of the asset relative to its total value for each ICSF and WCSF combination. This figure indicates that stochastic price variability contributes significantly to the total value of the asset (on average this contribution is 21% of this value).

Figure 11 also shows that the relative extrinsic values are “V” shaped in WCSF and they do not necessarily increase in ICSF. These behaviors can be explained as follows. It is clear that both the total value and the intrinsic value of the asset increase in ICSF and WCSF. In fact, as illustrated in Figures 12 and 13, these values per unit of available space rise at a decreasing rate in these quantities. However, these values exhibit different rates of increase. As a consequence, their
Figure 12: Total values of the asset per unit of available space for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow.

Figure 13: Intrinsic values of the asset per unit of available space for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow.
Figure 14: Extrinsic values of the asset per unit of available space for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow when WCSF is equal to or greater than 0.3.

differences, that is, the extrinsic values of the asset per unit of available space, can increase at a slower rate than the intrinsic value or can even decrease. Figure 14 displays the extrinsic values of the asset per unit of available space and shows that both of these cases can occur. In particular, when ICSF is less than or equal to 0.3 these values first decrease and then increase in WCSF. Moreover, when WCSF is less than or equal to 0.2 these values first increases and then decreases in ICSF.

**Capturing the value of price uncertainty.** Having established that price uncertainty can add substantial value to the intrinsic value of the asset, it is natural to question whether this incremental value could be seized (in expectation) by computing a simpler policy that SCOP. A natural policy to consider for this purpose is the reoptimized intrinsic value policy (RIVP), which can be sequentially constructed as follows (RIVP is essentially a certainty equivalent policy; see Bertsekas 2005 [2, Chapter 6]).

In each stage and state, one solves an MDP that only considers the current forward curve and implements the trade corresponding to the current stage and state. This process is then repeated in the next stage in the state obtained by observing the newly realized forward curve and by performing the stated trade in the current stage and state. In other words, price uncertainty is incorporated in RIVP by revising one’s decisions in a reactive fashion as opposed to the proactive
and forward looking manner in which SCOP is constructed. In this study, RIVP is implemented by taking the “realized” forward curve in a given stage and state to be the risk neutral expectation of the spot price in each of the remaining stages on the trinomial lattice conditional on the spot price in the given stage and state. RIVP can be easily computed and evaluated. The performance of RIVP is remarkable. Averaging across all the considered WCSF and ICSF cases the ratio of the RIVP and SCOP total values of the asset is 99.81%, and the minimum and maximum values of this ratio are 96.00% and 100.00%, respectively.

This suggests that in practice one might be able to make very good trading and operational decisions without explicitly modeling the uncertainty in the natural gas price evolution. However, this does not imply that this modeling is not relevant in practice. As discussed in §1, storage asset valuation is important in practice. Given that the magnitude of the extrinsic value of natural gas storage assets appears to be substantial, modeling price uncertainty when valuing such an asset is clearly relevant.

5 Conclusions

This paper studies the merchant management of capacitated commodity storage assets. Different from the existing literature on the warehouse problem, the version of the problem considered in this paper features both space and capacity limits. These limits are shown to interact in a nontrivial manner and to give rise to an optimal trading policy that at each decision time depends both on the realized spot price and the merchant’s available inventory, whereas in the traditional warehouse problem this structure only depends on the realized spot price. In other words, with a capacitated commodity storage asset the qualification of low (buying) and high (selling) prices depends on the merchant’s own inventory availability. The optimal policy structure is attractive, being fully characterized by two stage and spot price dependent basestock targets that, under some assumptions, partition the available inventory and spot price space into three BI/DN/WS regions. In some cases of practical importance, these targets can be easily computed.

The nature of the optimal policy structure implies that the merchant’s operational and trading decisions are linked. This paper investigates the likely practical relevance of coordinating these choices in the context of natural gas storage by using real data. Here, it shows that ignoring the interface between operational and trading decisions can yield significant losses.

This paper also quantifies the likely contribution to the total value of the asset of adapting the merchant’s trading decisions to the uncertain unfolding of natural gas prices. The value of this contribution is shown to be noteworthy and can be essentially captured by reacting to the
unfolding of price uncertainty; that is, by sequentially reoptimizing a model that only considers the seasonal (deterministic) component of the variability in future spot prices. Modeling price uncertainty remains relevant for the purposes of storage asset valuation, which is important in practice.

The analysis of this paper is based on the assumption that the spot price evolves according to a one factor model, in particular a seasonal mean reverting model in §4. While this model exhibits appealing features and is widely employed in the literature, additional research could focus on extending the analysis of this paper to other models of commodity price evolution, such as those of Routledge et al. (2000 [31]) and Schwartz and Smith (2000 [34]). In addition, it would be interesting to study the managerial issues investigated in this paper in other commodity settings, such as the storage of petroleum and petrochemical products, metals, and agricultural products.

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References


Online Appendix

A Additional Proofs

This section presents the proofs of the results stated in §3, except the proof of Theorem 1 that is given in §3.1.

A.1 Proofs for §3.1

Lemmas 1-2 are used in the proof of Proposition 1.

Lemma 1. Define

\[ V_j(s) := \sum_{k=j}^J \delta_i^{j,k} \left( \alpha^I E[\tilde{s}_k|\tilde{s}_j = s] + c^I \vee c^W + h \right), \forall j \in J, s \in S_j. \] (19)

It holds that \(|V_j(x, s)| \leq V_j(s), \forall j \in J, (x, s) \in X \times S_j. \)

Proof. By induction. In stage \(J\), the claimed property holds for each pair \((x, s) \in X \times S_j\) because

\[
|V_J(x, s)| = |(\alpha^W s - c^W)^+[(\alpha^W s - c^W) \land (x - x)] - hx| \\
\leq |(\alpha^W s - c^W)^+(x - x)| + |hx| \\
\leq (\alpha^I s + c^I \vee c^W)(x - x) + hx \\
\leq \bar{\alpha}(\alpha^I s + c^I \vee c^W + h) \\
= V_J(s).
\]

Make the induction hypothesis that the claimed property also holds in stages \(j+1, \ldots, J-1\). Let \(a^*_j(x, s)\) be an optimal action in stage \(j\) and feasible state \((x, s) \in X \times S_j\). Notice that

\[
|V_j(x, s)| = |p_j(a^*_j(x, s), s) - hx + \delta_j E_j[V_{j+1}(x + a^*_j(x, s), \tilde{s}_{j+1})]| \\
\leq |p_j(a^*_j(x, s), s)| + |hx| + |\delta_j E_j[V_{j+1}(x + a^*_j(x, s), \tilde{s}_{j+1})]| \\
\leq \bar{\alpha}(\alpha^I s + c^I \vee c^W + h) + |\delta_j E_j[V_{j+1}(x + a^*_j(x, s), \tilde{s}_{j+1})]| \\
\leq \bar{\alpha}(\alpha^I s + c^I \vee c^W + h) + \delta_j E_j \left| V_{j+1}(x + a^*_j(x, s), \tilde{s}_{j+1}) \right| \\
\leq \bar{\alpha}(\alpha^I s + c^I \vee c^W + h) \\
+ \delta_j E_j \left[ \bar{\alpha} \sum_{k=j+1}^J \delta_i^{j+1,k} \left( \alpha^I E[\tilde{s}_k|\tilde{s}_{j+1}] + c^I \vee c^W + h \right) \right] \\
= \bar{\alpha}(\alpha^I s + c^I \vee c^W + h)
\]

OA-1
\[
\sum_{k=j+1}^J \delta^{j+1,k} E \left[ \alpha^I \tilde{s}_k + c^I \lor c^W + h \big| \tilde{s}_j = s \right] + \pi \delta_j \sum_{k=j+1}^J \delta^{j+1,k} E \left[ \alpha^I \tilde{s}_k + c^I \lor c^W + h \big| \tilde{s}_j = s \right] \\
= \pi \sum_{k=j}^J \delta^{j,k} \left( \alpha^I E[\tilde{s}_k | \tilde{s}_j = s] + c^I \lor c^W + h \right) \\
= \nabla_j(s),
\]

so that the claimed property holds in every state in stage \( j \). Thus, this property holds in every stage and state by the principle of mathematical induction. \( \Box \)

**Lemma 2.** If Assumption 1 is true, then it holds that \( \delta_j E \left[ |V_{j+1}(x, \tilde{s}_{j+1})| \big| \tilde{s}_j = s \right] < \infty, \forall j \in J \setminus \{ J \}, (x, s) \in \mathcal{X} \times \mathcal{S}_j \).

**Proof.** Assumption 1 implies that the function \( \nabla_{j+1}(\cdot) \) defined by (19) satisfies

\[
\delta_j E \left[ \nabla_{j+1}(\tilde{s}_{j+1}) \big| \tilde{s}_j = s \right] = \delta_j \pi \sum_{k=j+1}^J \delta^{j+1,k} \left( \alpha^I E[\tilde{s}_k | \tilde{s}_{j+1} = s] + c^I \lor c^W + h \right) \\
= \delta_j \pi \sum_{k=j+1}^J \delta^{j+1,k} \left( \alpha^I E[\tilde{s}_k | \tilde{s}_j = s] + c^I \lor c^W + h \right) \\
< \infty, \forall j \in J \setminus \{ J \}, s \in \mathcal{S}_j.
\]

Lemma 1 and this inequality imply that

\[
\delta_j E \left[ |V_{j+1}(x, \tilde{s}_{j+1})| \big| \tilde{s}_j = s \right] < \delta_j E \left[ \nabla_{j+1}(\tilde{s}_{j+1}) \big| \tilde{s}_j = s \right] < \infty, \forall j \in J \setminus \{ J \}, (x, s) \in \mathcal{X} \times \mathcal{S}_j. \Box
\]

**Proof of Proposition 1 (Concavity).** By induction. The claimed property clearly holds in stage \( J \). Make the induction hypothesis that this property also holds in stages \( j+1, \ldots, J-1 \). Consider stage \( j \). Pick \( \phi \in [0,1] \) and \( x^i \in \mathcal{X}, i = 1, 2 \), and define \( x^\phi := \phi x^1 + (1 - \phi)x^2 \), which is clearly in set \( \mathcal{X} \). The concavity of \( V_{j+1}(x, \tilde{s}_{j+1}) \) in \( x \), given \( s_{j+1} \), means that

\[
V_{j+1}(x^\phi, s_{j+1}) \geq \phi V_{j+1}(x^1, s_{j+1}) + (1 - \phi)V_{j+1}(x^2, s_{j+1}). \tag{20}
\]

Assumption 1 and Lemmas 2 imply that

\[
U_j(x, s) = \delta_j E \left[ V_{j+1}(x, \tilde{s}_{j+1}) | \tilde{s}_j = s \right] \leq \delta_j E_j \left[ |V_{j+1}(x, \tilde{s}_{j+1})| \big| \tilde{s}_j = s \right] < \infty, \forall (x, s) \in \mathcal{X} \times \mathcal{S}_j, \tag{21}
\]

so that \( U_j(x, s) \) is real valued in \( x \in \mathcal{X} \) for each given \( s \in \mathcal{S}_j \). Discounting and taking expectations on both sides of (20) yields

\[
U_j(x^\phi, s) \geq \phi U_j(x^1, s) + (1 - \phi)U_j(x^2, s),
\]

OA-2
so that \( U_j(x, s) \) is concave in \( x \in \mathcal{X} \) for each given \( s \in \mathcal{S}_j \).

Let \( a^i \) be a feasible action at inventory level \( x^i, \ i = 1, 2, \) and define \( a^\phi := \phi a^1 + (1 - \phi) a^2 \). The convexity of the inventory action set \( \mathcal{C} \), defined in §2, implies that \( (x^\phi, a^\phi) \in \mathcal{C} \). The concavity of \( U_j(x, s) \) in \( x \) given \( s \) implies that
\[
U_j(x^\phi + a^\phi, s) \geq \phi U_j(x^1 + a^1, s) + (1 - \phi) U_j(x^2 + a^2, s).
\]

The immediate payoff function \( p_j(a, s) \) is piecewise linear and concave in \( a \in \mathcal{R} \) for each given \( s \in \mathcal{S}_j \). This, the linearity of \( -hx \) in \( x \in \mathcal{X} \), and (22) imply that \( v_j(x, a, s) \) is jointly concave in \((x, a) \in \mathcal{C} \) for each given \( s \in \mathcal{S}_j \). By Proposition B-4 in Heyman and Sobel (2004 [51, p. 525]), \( V_j(x, s) \) is concave in \( x \in \mathcal{X} \) for each given \( s \in \mathcal{S}_j \). Thus, the claimed property holds in stage \( j \). By the principle of mathematical induction this property holds in every stage. □

### A.2 Proofs for §3.2

Lemmas 3-7 establish simple but useful properties that are used to prove Theorem 2.

**Lemma 3.** If Assumption 2 holds then \( \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s] - s \) decreases in \( s \in \mathcal{S}_j, \forall j \in \mathcal{J} \setminus \{J\} \).

**Proof.** Pick stage \( j \in \mathcal{J} \setminus \{J\} \). From (a) of Assumption 2 and Corollary 3.9.1(a) in Topkis (1998 [53]) imply that the function
\[
\delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s] = \delta_j (\alpha^I - 1) E[\tilde{s}_{j+1} | \tilde{s}_j = s]
\]
increases in \( s \in \mathcal{S}_j \). Thus, for \( s', s'' \in \mathcal{S}_j \) with \( s' < s'' \) it holds that
\[
\delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s'] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s'] \leq \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s''],
\]
or, equivalently,
\[
\delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s'] \leq \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s'].
\]

Part (b) of Assumption 2 means that
\[
\delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s'] - \alpha^W s' \geq \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s''] - \alpha^W s'',
\]
or, equivalently,
\[
\alpha^W (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s'].
\]

It now follows that
\[
s'' - s' \geq \alpha^W (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} | \tilde{s}_j = s'] \geq \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s'] \geq \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s'] \geq \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s''] - \delta_j E[\tilde{s}_{j+1} | \tilde{s}_j = s'] \geq (23),
\]

OA-3
which implies that
\[ \delta_j E[\bar{s}_{j+1} | \bar{s}_j = s'] - s' \geq \delta_j E[\bar{s}_{j+1} | \bar{s}_j = s''] - s''. \] \[
\]
Lemma 4. If Assumption 2 holds then \( \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s] - (\alpha^l s + c^l) \) decreases in \( s \in S_j \), \( \forall j \in J \setminus \{J\} \).

Proof. Pick stage \( j \in J \setminus \{J\} \). For \( s', s'' \in S_j \) with \( s' < s'' \), Lemma 3, which holds by Assumption 2, implies the inequality
\[ \delta_j E[\alpha^l \bar{s}_{j+1} | \bar{s}_j = s'] + \delta_j c^l - \alpha^l s' - c^l \geq \delta_j E[\alpha^l \bar{s}_{j+1} | \bar{s}_j = s''] + \delta_j c^l - \alpha^l s'' - c^l, \]
which can be rearranged as
\[ \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s'] - (\alpha^l s' + c^l) \geq \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s''] - (\alpha^l s'' + c^l). \]

Lemma 5. If Assumption 2 holds then \( \delta_j E[\alpha^W \bar{s}_{j+1} + c^W | \bar{s}_j = s] - (\alpha^l s + c^l) \) decreases in \( s \in S_j \), \( \forall j \in J \setminus \{J\} \).

Proof. Similar to the proof of Lemma 4. \( \square \)

Lemma 6. Suppose that Assumption 2 holds. Fix stage \( j \in J \setminus \{J\} \). Assume that there exists a function \( g : S_{j+1} \to \mathbb{R} \) that increases in \( s_{j+1} \in S_{j+1} \) and satisfies \( g(s_{j+1}) \leq \alpha^l s_{j+1} + c^l \), \( \forall s_{j+1} \in S_{j+1} \). (a) The function \( \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s] - (\alpha^l s + c^l) \) decreases in \( s \in S_j \). (b) The function \( \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s] - (\alpha^W s + c^W) \) decreases in \( s \in S_j \).

Proof. The second assumption made on \( g \) implies that
\[ \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s] \leq \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s], \forall s \in S_j. \] (25)
The first assumption made on \( g \), part (a) of Assumption 2, and Corollary 3.9.1(a) in Topkis (1998 [53]) imply that
\[ \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s] \text{ increases in } s \in S_j; \] (26) part (a) of Assumption 2 and Corollary 3.9.1(a) in Topkis (1998 [53]) imply that
\[ \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s] \text{ increases in } s \in S_j. \] (27)
It follows from (25)-(27) that \( \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s] - \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s] \) increases in \( s \in S_j \). Letting \( s', s'' \in S_j \) with \( s' < s'' \), this means that
\[ \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s''] - \delta_j E[g(\bar{s}_{j+1}) | \bar{s}_j = s'] \leq \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s''] - \delta_j E[\alpha^l \bar{s}_{j+1} + c^l | \bar{s}_j = s'], \] (28)

OA-4
Focus on the property claimed in part (a). Assumption 2 and Lemma 4 imply that \( \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s] - (\alpha^I s + c^I) \) decreases in \( s \in S_j \):

\[
\alpha^I (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s'].
\] (29)

Combining (28)-(29) yields

\[
\alpha^I (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s']
\]

\[
\alpha^I (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s'] - \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s'].
\] (30)

Part (a) follows because these inequalities imply that

\[
\delta_j E[g(\tilde{s}_{j+1}) | \tilde{s}_j = s'] - (\alpha^I s' + c^I) \geq \delta_j E[g(\tilde{s}_{j+1}) | \tilde{s}_j = s''] - (\alpha^I s'' + c^I).
\]

Consider the property claimed in part (b). Part (b) of Assumption 2 is equivalent to stating that \( \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s] - (\alpha^W s - c^W) \) decreases in \( s \in S_j \):

\[
\alpha^W (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s'].
\] (31)

Combining (28)-(31) yields

\[
\alpha^W (s'' - s') \geq \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s''] - \delta_j E[\alpha^I \tilde{s}_{j+1} + c^I | \tilde{s}_j = s']
\]

\[
\alpha^W (s'' - s') \geq \delta_j E[g(\tilde{s}_{j+1}) | \tilde{s}_j = s''] - \delta_j E[g(\tilde{s}_{j+1}) | \tilde{s}_j = s'],
\]

Part (b) holds because these inequalities can be rearranged as

\[
\delta_j E[g(\tilde{s}_{j+1}) | \tilde{s}_j = s'] - (\alpha^W s' - c^W) \geq \delta_j E[g(\tilde{s}_{j+1}) | \tilde{s}_j = s''] - (\alpha^W s'' - c^W). \]

**Lemma 7.** Define

\[
V_{j+1}(x, s) := \lim_{\epsilon \downarrow 0} \frac{V_{j+1}(x, s) - V_{j+1}(x, s - \epsilon, s)}{\epsilon}, \quad \forall j \in J \setminus \{J\}, \quad x \in X^o, \quad s \in S_{j+1}
\] (31)

\[
V_{j+1}(x, s) := \lim_{x \uparrow \overline{x}} \frac{V_{j+1}(x, s)}{\epsilon}, \quad \forall j \in J \setminus \{J\}, \quad \forall s \in S_{j+1}
\]

\[
V_{j+1}(x, s) := \lim_{x \downarrow \underline{x}} \frac{V_{j+1}(x, s)}{\epsilon}, \quad \forall j \in J \setminus \{J\}, \quad \forall s \in S_{j+1}.
\] (33)

It holds that

\[
U_j'(x, s, \tilde{s}_j) = \delta_j E \left[ V_{j+1}(x, \tilde{s}_{j+1}) | \tilde{s}_j = s \right], \quad \forall j \in J \setminus \{J\}, \quad (x, s) \in X \times S_j.
\] (34)
\textbf{Proof.} It is not hard to show that in each stage \( j \in \mathcal{J} \) and for each \( s \in \mathcal{S}_j \) the function \( V_j(x, s) \) is Lipschitz continuous in \( x \in \mathcal{X} \) with constant \( K_j(s) := \sum_{k=j}^J \delta_{j,k} \left( \alpha^j \mathbb{E}[\delta s_j | \bar{s}_j = s] + c^j \vee c^W + h \right): \)

\[ |V_j(x^2, s) - V_j(x^1, s)| \leq K_j(s)|x^2 - x^1|, \quad \forall x^1, x^2 \in \mathcal{X}. \tag{35} \]

(Notice that \( K_j(s) < \infty \) by Assumption 1.)

Fix stage \( j \in \mathcal{J} \setminus \{J\}, s \in \mathcal{S}_j \), and \( z \in \mathcal{S}_{j+1} \) (the notation \( z \) in lieu of \( s_{j+1} \) is used for expositional convenience). Pick \( x \in \mathcal{X}^o \). For each positive \( \epsilon \) such that \( x - \epsilon \in \mathcal{X} \), (35) implies that

\[ \frac{|V_{j+1}(x, z) - V_{j+1}(x - \epsilon, z)|}{\epsilon} \leq K_{j+1}(z). \]

Moreover, by Assumption 1, it holds that \( E[K_{j+1}(\bar{s}_{j+1})|\bar{s}_j = s] < \infty \). Similar to the proof of Lemma 3.2 in Glasserman and Tayur (1995 [50]), it follows from the Dominated Convergence Theorem (see, e.g., Resnick 1999 [52, p. 133]) that

\[ U'_j(x, s) \equiv \lim_{\epsilon \downarrow 0} \frac{U_j(x, s) - U_j(x - \epsilon, s)}{\epsilon} = \delta_j \lim_{\epsilon \downarrow 0} E \left[ \frac{V_{j+1}(x, \bar{s}_{j+1}) - V_{j+1}(x - \epsilon, \bar{s}_{j+1})}{\epsilon} \mid \bar{s}_j = s \right], \]

and (34) holds for \( x \in \mathcal{X}^o \).

Pick \( x = \bar{x} \). Proposition 1 and (35) imply that

\[ U'_j(\bar{x}, s) \equiv \lim_{x \downarrow \bar{x}} U'_j(x, s) = \lim_{\epsilon \downarrow 0} \frac{U_j(\bar{x} + \epsilon, s) - U_j(\bar{x}, s)}{\epsilon} = \delta_j \lim_{\epsilon \downarrow 0} E \left[ \frac{V_{j+1}(\bar{x} + \epsilon, \bar{s}_{j+1}) - V_{j+1}(\bar{x}, \bar{s}_{j+1})}{\epsilon} \mid \bar{s}_j = s \right], \]

and (34) for \( x = \bar{x} \) follows from an application of the Dominated Convergence Theorem.

Finally, pick \( x = \bar{\pi} \). Proposition 1 and (35) imply that

\[ U'_j(\bar{\pi}, s) \equiv \lim_{x \downarrow \bar{\pi}} U'_j(x, s) = \lim_{\epsilon \downarrow 0} \frac{U_j(\bar{\pi} + \epsilon, s) - U_j(\bar{\pi}, s)}{\epsilon} = \delta_j \lim_{\epsilon \downarrow 0} E \left[ \frac{V_{j+1}(\bar{\pi} + \epsilon, \bar{s}_{j+1}) - V_{j+1}(\bar{\pi}, \bar{s}_{j+1})}{\epsilon} \mid \bar{s}_j = s \right], \]

and (34) for \( x = \bar{\pi} \) also follows from an application of the Dominated Convergence Theorem. \( \square \)
Proof of Theorem 2 (Price monotonicity). By induction. The property holds in stage $J$ because it is easy to verify that $b_J(s) = x$ and $\bar{b}_J(s) = \bar{x}$ if $s < cW / \alpha W$ and $b_J(s) = \bar{b}_J(s) = x$ otherwise. Make the first induction hypothesis that it also holds in stages $j + 1, \ldots, J - 1$.

Notice that $U'_{j}(x, s) = 0$, so that it trivially increases in $s \in S_J$ for each given $x \in \mathcal{X}$. Make the second induction hypothesis that the function $U'_{k}(x, s)$ increases in $s \in S_k$ for each given $x \in \mathcal{X}$ in every stage $k = j + 1, \ldots, J - 1$.

Moreover, it holds that the functions $U'_{j}(x, s) - (\alpha I_s + c^I) = -(\alpha I_s + c^I)$ decreases in $s \in S_J$ for each given $x \in \mathcal{X}$. Make the third induction hypothesis that the functions $U'_{k}(x, s) - (\alpha W s - c^W)$ decrease in $s \in S_k$ for each given $x \in \mathcal{X}$ in every stage $k = j + 1, \ldots, J - 1$.

Consider stage $j$. Given $s \in S_J$, the optimal basestock targets $b_J(s)$ and $\bar{b}_J(s)$, respectively, are optimal solutions to (12) and (13), which are displayed here for convenience:

\[
\max_{y \in \mathcal{X}} U_j(y, s) - (\alpha I_s + c^I)y,
\]

\[
\max_{y \in \mathcal{X}} U_j(y, s) - (\alpha W s - c^W)y.
\]

By Proposition 1, the objective functions of these maximizations are concave in the decision variable $y$. It is easy to see that $b_J(s)$ and $\bar{b}_J(s)$ satisfy the claimed property if

\[
U'_{j}(y, s) - (\alpha I_s + c^I) \text{ decreases in } s \text{ for each given } y \in \mathcal{X}
\]

\[
U'_{j}(y, s) - (\alpha W s - c^W) \text{ decreases in } s \text{ for each given } y \in \mathcal{X}.
\]

This will now be shown to be the case. Pick $(x, s) \in \mathcal{X} \times S_J$ and consider the function $U_j(x, s)$, which recall is defined to be $\delta_j E[V_{j+1}(x, \tilde{s}_{j+1}) | s_j = s]$. Focus on the function $V_{j+1}(x, z)$ in feasible state $(x, z)$ in stage $j + 1$. Consider the optimal action in this state and stage. Theorem 1 implies that only the following mutually exclusive cases need to be considered:

1. WS is optimal but $\delta_{j+1}(z)$ cannot be reached from $x$; that is, only $x + C$ can be reached from $x$;

2. WS is optimal and $\delta_{j+1}(z)$ can be reached from $x$;

3. DN is optimal;

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Accordingly, define the following mutually exclusive events:

- \( A^1_{j+1}(x, z) := \{ x + C > b_{j+1}(z) \} \),
- \( A^2_{j+1}(x, z) := \{ x + C < b_{j+1}(z), \ x > b_{j+1}(z) \} \),
- \( A^3_{j+1}(x, z) := \{ b_{j+1}(z) \leq x \leq b_{j+1}(z) \} \),
- \( A^4_{j+1}(x, z) := \{ x + C > b_{j+1}(z), \ x < b_{j+1}(z) \} \), and
- \( A^5_{j+1}(x, z) := \{ x + C < b_{j+1}(z) \} \).

Notice that for state \((x, z)\) it holds that \( \sum_{\ell=1}^{5} 1 \left\{ A^\ell_{j+1}(x, z) \right\} = 1 \), where \( 1\{\cdot\} \) equals 1 if its argument is true and 0 otherwise. Hence, \( V_{j+1}(x, z) \) can be written as

\[
V_{j+1}(x, z) = \left[ -(a^W z - c^W)[C + U_{j+1}(x + C, z)] \right] 1 \left\{ A^1_{j+1}(x, z) \right\} \\
+ \left[ -(a^W z - c^W)[b_{j+1}(z) - x] + U_{j+1}(b_{j+1}(z), z) \right] 1 \left\{ A^2_{j+1}(x, z) \right\} \\
+ U_{j+1}(x, z) 1 \left\{ A^3_{j+1}(x, z) \right\} \\
+ \left[ -(a^I z + c^I)[b_{j+1}(z) - x] + U_{j+1}(b_{j+1}(z), z) \right] 1 \left\{ A^4_{j+1}(x, z) \right\} \\
+ \left[ -(a^I z + c^I)C + U_{j+1}(x + C, z) \right] 1 \left\{ A^5_{j+1}(x, z) \right\}. \tag{40}
\]

Consider the function \( V'_{j+1}(x, z) \) defined by (31)-(33). It follows from (40) that

\[
V'_{j+1}(x, z) = U'_{j+1}(x + C, z) \left\{ A^1_{j+1}(x, z) \right\} + (a^W z - c^W) 1 \left\{ A^2_{j+1}(x, z) \right\} \\
+ U'_{j+1}(x, z) 1 \left\{ A^3_{j+1}(x, z) \right\} \\
+ (a^I z + c^I) 1 \left\{ A^4_{j+1}(x, z) \right\} + U'_{j+1}(x + C, z) 1 \left\{ A^5_{j+1}(x, z) \right\}. \tag{41}
\]

By adding and subtracting to the right hand side of (41) the quantities

\[
(a^W z - c^W) \left[ 1 \left\{ A^1_{j+1}(x, z) \right\} + 1 \left\{ A^2_{j+1}(x, z) \right\} \right], \\
(a^I z + c^I) \left[ 1 \left\{ A^3_{j+1}(x, z) \right\} + 1 \left\{ A^4_{j+1}(x, z) \right\} \right],
\]

expression (41) can be rearranged as

\[
V'_{j+1}(x, z) = \left[ U'_{j+1}(x + C, z) - (a^W z - c^W) \right] 1 \left\{ A^1_{j+1}(x, z) \right\}
\]
\[ + \left[ \left( \alpha^W z - c^W \right) - \left( \alpha^W z - c^W \right) \right] 1 \left\{ A^2_{j+1}(x, z) \right\} \\
+ \left[ \left( \alpha^I z + c^I \right) - \left( \alpha^I z + c^I \right) \right] 1 \left\{ A^4_{j+1}(x, z) \right\} \\
+ \left[ U'_{j+1}(x + \mathcal{C}, z) - \left( \alpha^I z + c^I \right) \right] 1 \left\{ A^5_{j+1}(x, z) \right\} \\
+ (\alpha^W z - c^W) \left[ 1 \left\{ A^1_{j+1}(x, z) \right\} + 1 \left\{ A^2_{j+1}(x, z) \right\} \right] \\
+ U'_{j+1}(x, z) 1 \left\{ A^3_{j+1}(x, z) \right\} \\
+ (\alpha^I z + c^I) \left[ 1 \left\{ A^4_{j+1}(x, z) \right\} + 1 \left\{ A^5_{j+1}(x, z) \right\} \right]. \tag{42} \]

To reduce the notational burden, define the following functions:

\[
\begin{align*}
  f^1_{j+1}(x, z) & := \left[ U'_{j+1}(x + \mathcal{C}, z) - (\alpha^W z - c^W) \right] 1 \left\{ A^1_{j+1}(x, z) \right\} \\
  & + 0 \cdot \left( 1 \left\{ A^2_{j+1}(x, z) \right\} + 1 \left\{ A^3_{j+1}(x, z) \right\} + 1 \left\{ A^4_{j+1}(x, z) \right\} \right) \\
  & + \left[ U'_{j+1}(x + \mathcal{C}, z) - (\alpha^I z + c^I) \right] 1 \left\{ A^5_{j+1}(x, z) \right\} \tag{43} \\
  f^2_{j+1}(x, z) & := (\alpha^W z - c^W) \left[ 1 \left\{ A^1_{j+1}(x, z) \right\} + 1 \left\{ A^2_{j+1}(x, z) \right\} \right] \\
  & + U'_{j+1}(x, z) 1 \left\{ A^3_{j+1}(x, z) \right\} \\
  & + (\alpha^I z + c^I) \left[ 1 \left\{ A^4_{j+1}(x, z) \right\} + 1 \left\{ A^5_{j+1}(x, z) \right\} \right]. \tag{44}
\end{align*}
\]

Then, (42) can be expressed as

\[ V'_{j+1}(x, z) = f^1_{j+1}(x, z) + f^2_{j+1}(x, z). \tag{45} \]

To study the behavior of the functions \( f^1_{j+1}(x, z) \) and \( f^2_{j+1}(x, z) \) in \( z \) given \( x \), it is useful to make some preliminary observations. Consider the determination of an optimal action in state \((x, z)\) in stage \( j + 1 \) as the spot price \( z \) varies in \( S_{j+1} \). By the first induction hypothesis, there exist no more than four ordered prices that depend on \( x \), denoted, with a slight abuse of notation, by \( s^1_{j+1}(x) \), \( s^2_{j+1}(x) \), \( s^3_{j+1}(x) \), and \( s^4_{j+1}(x) \) with \( s^1_{j+1}(x) \leq s^2_{j+1}(x) \leq s^3_{j+1}(x) \leq s^4_{j+1}(x) \), that can be used to partition set \( S_{j+1} \) into the mutually exclusive and exhaustive sets

- \( S^1_{j+1}(x) := (s^1_{j+1}(x), \infty) \cap S_{j+1} \),
- \( S^2_{j+1}(x) := (s^2_{j+1}(x), s^1_{j+1}(x)) \cap S_{j+1} \),
- \( S^3_{j+1}(x) := (s^3_{j+1}(x), s^2_{j+1}(x)) \cap S_{j+1} \),
- \( S^4_{j+1}(x) := (s^4_{j+1}(x), s^3_{j+1}(x)) \cap S_{j+1} \),

and

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\( \mathcal{S}_{j+1}^5(x) := [0, s_{j+1}(x)) \cap S_{j+1}, \)

which satisfy the property that \( z \in \mathcal{S}_{j+1}^f(x) \) if and only if \( 1 \{ A_{j+1}^f(x, z) \} = 1 \) for all \( \ell \in \{1, 2, 3, 4, 5\} \).

Consider the three following cases: Optimal WS, Optimal BI, Optimal DN.

**Optimal WS.** If \( z \in \mathcal{S}_{j+1}^1(x) \cup \mathcal{S}_{j+1}^2(x) \), the relevant objective function is

\[
U_{j+1}(y, z) - (\alpha^W z - c^W) y + (\alpha^W z - c^W - h)x, \tag{46}
\]

where \( y \) is the inventory level to be reached in stage \( j + 2 \); that is, \( y \) is the decision variable in the maximization of (46) subject to the constraint that \( y \in [x, x] \). In the following, refer to the quantity

\[
U'_{j+1}(y, z) - (\alpha^W z - c^W) \tag{47}
\]

as the slope of (46) in \( y \). Observations 1-3 characterize the sign of (47).

**Observation 1.** If \( z \in \mathcal{S}_{j+1}^1(x) \), then \( \overline{b}_{j+1}(z) \) maximizes (46) but is not reachable from \( x \), so that the slope (47) at \( x + C \) satisfies

\[
U'_{j+1}(x + C, z) - (\alpha^W z - c^W) \leq 0, \quad \forall z \in \mathcal{S}_{j+1}^1(x). \tag{48}
\]

(In words, since the optimum \( \overline{b}_{j+1}(z) \) in the optimization of (46) is not reachable from \( x \), it must lie to the left of \( x + C \) and at this inventory level the objective function (46) cannot be strictly increasing in \( y \).)

**Observation 2.** If \( z \in \mathcal{S}_{j+1}^1(x) \cup \mathcal{S}_{j+1}^2(x) \), then \( \overline{b}_{j+1}(z) \) maximizes (46) so that the slope of this function evaluated at \( x \) is negative:

\[
U'_{j+1}(x, z) \leq \alpha^W z - c^W, \quad \forall z \in \mathcal{S}_{j+1}^2(x). \tag{49}
\]

**Observation 3.** If \( \mathcal{S}_{j+1}^1(x) \neq \emptyset \) and \( z \in \mathcal{S}_{j+1}^2(x) \), then \( \overline{b}_{j+1}(z) \) maximizes (46) and is reachable from \( x \), so that the slope (47) evaluated at \( x + C \), where \( x + C \in \mathcal{X} \) because \( \mathcal{S}_{j+1}^1(x) \neq \emptyset \) implies that \( \exists z' \in \mathcal{S}_{j+1}^1(x) \) such that \( 1 \{ A^1(x, z') \} = 1 \), satisfies

\[
U'_{j+1}(x + C, z) - (\alpha^W z - c^W) \geq 0, \quad \forall z \in \mathcal{S}_{j+1}^2(x). \tag{50}
\]

(In words, since the optimum \( \overline{b}_{j+1}(z) \) in the optimization of (46) is reachable from \( x \) and \( x + C > \overline{b}_{j+1}(z') \) for some \( z' \in \mathcal{S}_{j+1}^1 \), it holds that \( x < x + C \leq \overline{b}_{j+1}(z) \) and at \( x + C \) the objective function (46) is increasing in \( y \).)
**Optimal BI.** If \( z \in S_{j+1}^{4}(x) \cup S_{j+1}^{5}(x) \), the relevant objective function is

\[
U_{j+1}(y, z) - (\alpha^j z + c^j)y + (\alpha^j z + c^j - h)x,
\]

where \( y \) is the decision variable in the maximization of (51) subject to the constraint that \( y \in [x, \mathbf{x}] \).

In the following, refer to

\[
U'_{j+1}(y, z) - (\alpha^j z + c^j)
\]

as the slope of (51) in \( y \). Observations 4-6 characterize the sign of (52).

**Observation 4.** If \( z \in S_{j+1}^{4}(x) \cup S_{j+1}^{5}(x) \), \( b_{j+1}(z) \) maximizes (51) so that the slope of this function evaluated at \( x \) is positive:

\[
U'_{j+1}(x, z) \geq \alpha^j z + c^j, \forall z \in S_{j+1}^{4}(x).
\]

(53)

**Observation 5.** If \( S_{j+1}^{5}(x) \neq \emptyset \) and \( z \in S_{j+1}^{4}(x) \), then \( b_{j+1}(z) \) maximizes (51) and is reachable from \( x \), so that the slope (52) at \( x + \mathbf{C} \), where \( x + \mathbf{C} \in \mathcal{X} \) because \( S_{j+1}^{5}(x) \neq \emptyset \) implies that \( \exists z' \in S_{j+1}^{5}(x) \) such that \( 1 \{ A^5(x, z') \} = 1 \), satisfies

\[
U'_{j+1}(x + \mathbf{C}, z) - (\alpha^j z + c^j) \leq 0, \forall z \in S_{j+1}^{5}(x).
\]

(54)

(In words, since the optimum \( b_{j+1}(z) \) in the optimization of (51) is reachable from \( x \) and \( x + \mathbf{C} < b_{j+1}(z') \) for some \( z' \in S_{j+1}^{5}(x) \), it holds that \( x + \mathbf{C} \geq b_{j+1}(z) \) and at \( x + \mathbf{C} \) the objective function (51) is decreasing.)

**Observation 6.** If \( z \in S_{j+1}^{5}(x) \), then \( b_{j+1}(z) \) maximizes (51) and is not reachable from \( x \), so that the slope (52) at \( x + \mathbf{C} \) satisfies

\[
U'_{j+1}(x + \mathbf{C}, z) - (\alpha^j z + c^j) \geq 0, \forall z \in S_{j+1}^{5}(x).
\]

(55)

(In words, since the optimum \( b_{j+1}(z) \) in the optimization of (51) is not reachable from \( x \), it must lie to the right of \( x + \mathbf{C} \) and at this inventory level the objective function (51) cannot be strictly decreasing in \( y \).)

**Optimal DN.** Only Observation 7 is associated with this case.

**Observation 7.** If \( z \in S_{j+1}^{3}(x) \), the DN action is optimal at \( x \), which means that every other type of action is not, so that it holds that

\[
U'_{j+1}(x, z) \geq \alpha^W z - c^W
\]

\[
U'_{j+1}(x, z) \leq \alpha^j z + c^j,
\]

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which can be rearranged as
\[ \alpha^W z - c^W \leq U'_{j+1}(x, z) \leq \alpha^I z + c^I, \quad \forall z \in S^4_{j+1}(x). \] (56)

The previous observations are now used to determine the behaviors of the functions \( f^1_{j+1}(x, z) \) and \( f^2_{j+1}(x, z) \) in \( z \) with \( x \) kept fixed. It is useful to recall the property that \( z \in S^5_{j+1}(x) \) if and only if \( \{ A^\ell_{j+1}(x, z) \} = 1 \) for all \( \ell \in \{1, 2, 3, 4, 5 \} \).

Consider \( f^1_{j+1}(x, z) \), defined by (43). Given \( x \), this function is positive for \( z \in S^5_{j+1}(x) \) by (55), zero for \( z \in S^2_{j+1}(x) \cup S^3_{j+1}(x) \cup S^4_{j+1}(x) \) by definition, and negative for \( z \in S^1_{j+1}(x) \) by (48). Moreover, the third induction hypothesis implies that this function decreases in \( z \in S_{j+1} \). Part (a) of Assumption 2 and Corollary 3.9.1(a) in Topkis (1998 [53]) imply that
\[ \delta_j E[f^1_{j+1}(x, \tilde{s}_{j+1})|\tilde{s}_j = s] \text{ decreases in } s \in S_j. \] (57)

Consider \( f^2_{j+1}(x, z) \), defined by (44). Given \( x \), inequalities (49), (53), (56), and the second induction hypothesis imply that
\[ f^2_{j+1}(x, z) \text{ increases in } z \in S_{j+1}; \] (58)
\[ f^2_{j+1}(x, z) \leq \alpha^I z + c^I, \quad \forall z \in S_{j+1}. \] (59)

By Assumption 2 and properties (58)-(59), Lemma 6 can be applied to function \( f^2_{j+1}(x, z) \), which yields that
\[ \delta_j E[f^2_{j+1}(x, \tilde{s}_{j+1})|\tilde{s}_j = s] - (\alpha^I s + c^I) \text{ decreases in } s \in S_j; \] (60)
\[ \delta_j E[f^2_{j+1}(x, \tilde{s}_{j+1})|\tilde{s}_j = s] - (\alpha^W s - c^W) \text{ decreases in } s \in S_j. \] (61)

Lemma 7, definitions (31)-(33), and expression (45) imply that
\[ U'_j(x, s) - (\alpha^I s + c^I) = \delta_j E[f^1_{j+1}(\tilde{s}_{j+1})|\tilde{s}_j = s] + \delta_j E[f^2_{j+1}(\tilde{s}_{j+1})|\tilde{s}_j = s] - (\alpha^I s + c^I) \]
\[ U'_j(x, s) - (\alpha^W s - c^W) = \delta_j E[f^1_{j+1}(\tilde{s}_{j+1})|\tilde{s}_j = s] + \delta_j E[f^2_{j+1}(\tilde{s}_{j+1})|\tilde{s}_j = s] - (\alpha^W s - c^W). \]

Then, properties (57), (60), and (61) imply that both \( U'_j(x, s) - (\alpha^I s + c^I) \) and \( U'_j(x, s) - (\alpha^W s - c^W) \) decrease in \( s \in S_j \) for each given \( x \in X \). Hence, both conditions (38) and (39) hold and \( \tilde{b}_j(s) \) and \( \tilde{b}_j(s) \) satisfy the claimed property.

It remains to be shown that \( U'_j(x, s) \) increases in \( s \in S_j \) for each given \( x \in X \). Consider \( V'_{j+1}(x, z) \) as expressed in (41). The second induction hypothesis and inequalities (54)-(55) imply that the function
\[ U'_{j+1}(x + \mathcal{U}, z)1 \left\{ z \in S^5_{j+1}(x) \right\} + (\alpha^I z + c^I)1 \left\{ z \in S^4_{j+1}(x) \right\} \]
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increases in $z \in S_{j+1}^3(x) \cup S_{j+1}^4(x)$. The function
\[
(a'z + c')1 \{z \in S_{j+1}^4(x)\} + U'_{j+1}(x,z)1 \{z \in S_{j+1}^3(x)\} + (\alpha W s_j - c^W)1 \{z \in S_{j+1}^2(x)\}
\]
increases in $z \in S_{j+1}^4(x) \cup S_{j+1}^3(x) \cup S_{j+1}^2(x)$ because on this subset of $S_{j+1}$ this function is equal to $f_{j+1}(x,z)$, which has already been shown to increase in $z \in S_{j+1}$. The second induction hypothesis and inequalities (48) and (50) imply that the function
\[
U'_{j+1}(x + C, z)1 \{z \in S_{j+1}^2(x)\} + (\alpha W z - c^W)1 \{z \in S_{j+1}^1(x)\}
\]
increases in $z \in S_{j+1}^2(x) \cup S_{j+1}^1(x)$. Thus, $V'_{j+1}(x,z)$ increases in $z \in S_{j+1}$. Part (a) of Assumption 2 and Corollary 3.9.1(a) in Topkis (1998 [53]) imply that $U_{j}(x,s_j) = \delta_j E[V_{j+1}(x,z)|\bar{s}_j = s]$, which holds by Lemma 7, increases in $s \in S_j$.

Thus, the claimed property holds in every stage by the principle of mathematical induction. □

A.3 Proofs for §3.3

Proof of Proposition 2 (Piecewise linearity). The stated property clearly holds in stage $J$ because
\[
V_J(x,s) = -\left[hx + (\alpha W s - c^W)(\mathcal{C} \lor (x - x))1 \{s \geq c^W/\alpha W\}\right], \forall (x,s) \in \mathcal{X} \times S_J.
\]
Make the induction hypothesis that this property also holds in stages $j+1, \ldots, J - 1$. Consider stage $j$. Fix $s \in S_j$. Assumption 3 and the induction hypothesis imply that the function $U_j(x,s)$ is piecewise linear and continuous in $x \in \mathcal{X}$, because it is a weighted sum of a finite number of functions with this property where the weights are the terms $\delta_j \Pr\{\bar{s}_{j+1} = s_{j+1}|\bar{s}_j = s\}$ for each $s_{j+1} \in S_{j+1}$. To establish that the function $V_j(x,s)$ is also piecewise linear and continuous in $x \in \mathcal{X}$, Theorem 1 implies that there are three cases to consider: (BI) $x \in [x,b_j(s))$, (DN) $x \in [b_j(s),\bar{b}_j(s)]$, and (WS) $x \in (\bar{b}_j(s),\bar{x}]$. In the DN case, it holds that $V_j(x,s) = U_j(x,s) - hx$; in the BI case, if $b_j(s)$ is reachable from $x$ then it holds that $V_j(x,s) = U_j(b_j(s),s) - (\alpha' s + c')[b_j(s) - x] - hx$, otherwise it holds that $V_j(x,s) = U_j(x + \mathcal{C},s) - (\alpha' s + c')\mathcal{C} - hx$; in the WS case, if $\bar{b}_j(s)$ is reachable from $x$ then it holds that $V_j(x,s) = U_j(\bar{b}_j(s),s) - (\alpha W s - c^W)[\bar{b}_j(s) - x] - hx$, otherwise it holds that $V_j(x,s) = U_j(x + \mathcal{C},s) - (\alpha W s - c^W)\mathcal{C} - hx$. Thus, it is easy to check that the claimed property holds in stage $j$. By the principle of mathematical induction, this property holds in every stage. □
Proof of Proposition 3 (Restricted capacities and inventory limits). By induction. In stage $J$, the property holds because if $s < c \alpha W$, then $V_J(x, s) = -hx$, $\bar{b}_J(s) = \bar{x}$, and $\underline{b}_J(s) = \underline{x}$; otherwise $V_J(x, s) = -hx + (\alpha W s - c \alpha W)(-C \land (x - \underline{x}))$, which changes slope in inventory at $-C$, and $\bar{b}_J(s) = \bar{b}_J(s) = \bar{x}$. Make the induction hypothesis that the two claimed properties also hold in stages $j + 1, \ldots, J - 1$. Consider stage $j$. Fix $s \in S_j$. By Theorem 1, there are three relevant cases to consider: (BI) $x \in [\underline{x}, \underline{b}_j(s))$, (DN) $x \in [\underline{b}_j(s), \bar{b}_j(s)]$, and (WS) $x \in (\bar{b}_j(s), \bar{x}]$. In the DN case, $V_J(x, s) = U_J(x, s) - hx$; in the BI case, if $\underline{b}_j(s)$ can be reached from $x$ then $V_J(x, s) = U_J(\underline{b}_j(s), s) - (\alpha^I s + c^I)[\underline{b}_j(s) - x] - hx$, otherwise $V_J(x, s) = U_J(x + C, s) - (\alpha^I s + c^I)C - hx$; in the WS case, if $\bar{b}_j(s)$ is reachable from $x$ then $V_J(x, s) = U_J(\bar{b}_j(s), s) - (\alpha W s - c W)[\bar{b}_j(s) - x] - hx$, otherwise $V_J(x, s) = U_J(x + C, s) - (\alpha W s - c W)C - hx$. Thus, it is easy to check that property (a) holds in stage $j$. Consider property (b). The quantity $\underline{b}_j(s)$ can be determined by solving (12), where the objective function $U_J(y, s) - (\alpha^I s + c^I)y$ is to be maximized by choosing $y \in \mathcal{X}$. Proposition 2, which holds by Assumption 3, and the induction hypothesis imply that $U_J(y, s)$ can change slope in $y$ only at inventory levels that are integer multiples of $Q$. By the linearity in $y$ of the term $-(\alpha^I s + c^I)y$, the objective function of (12) also satisfies this property. Thus, $\underline{b}_j(s)$ can be taken to be an integer multiple of $Q$. An analogous argument can be used to show that $\bar{b}_j(s)$ can be chosen to satisfy the same property. Thus, property (b) holds in stage $j$. The principle of mathematical induction implies that the claimed properties hold in every stage. □

B  Numerical Values Displayed in Figures 7-14

This section reports the numerical values displayed in Figures 7-14. Tables 3-10 are associated with Figures 7-14 as follows:

- Table 3: Figure 7,
- Table 4: Figure 8,
- Table 5: Figure 9,
- Table 6: Figure 10,
- Table 7: Figure 11,
- Table 8: Figure 12,
- Table 9: Figure 13,
- Table 10: Figure 14.
Table 4: Gains on the total value of the asset of FCOP relative to SCOP for different ICSFs and WCSFs (Percent).

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<thead>
<tr>
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<th>Price</th>
<th>Maturity</th>
<th>Price</th>
</tr>
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</tr>
<tr>
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</tr>
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</tr>
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</tr>
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</tr>
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<td>11.443</td>
<td>23</td>
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Table 3: NYMEX forward curve on 2/1/2006 ($/MMBTU).

<table>
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</tr>
</thead>
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</tr>
<tr>
<td>24</td>
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</table>

Table 5: Flow rate gains of FCOP relative to SCOP for different ICSFs and WCSFs (Percent).

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<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
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<th>0.70</th>
<th>0.80</th>
<th>0.90</th>
<th>1.00</th>
</tr>
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<tr>
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Average = 18.18

OA-15
Table 6: Losses on the total value of the asset of DOTP relative to SCOP for different ICSFs and WCSFs (Percent).

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</tr>
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<td>16.33</td>
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<tr>
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Average = 76.58

Table 7: Relative extrinsic values of the asset for different ICSFs and WCSFs (Percent).

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Average = 20.76

Table 8: Total values of the asset per unit of available space for different ICSFs and WCSFs ($/MMBTU).

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Average = 3.80

OA-16
Table 9: Intrinsic values of the asset per unit of available space for different ICSFs and WCSFs ($/MMBTU).

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Average = 3.01

Table 10: Extrinsic values of the asset per unit of available space for different ICSFs and WCSFs ($/MMBTU).

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Average = 0.79
References


