Optimal Energy Procurement in Spot and Forward Markets

Nicola Secomandi, Sunder Kekre
Tepper School of Business, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213-3890, USA, {ns7, sk0a}@andrew.cmu.edu

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Abstract

Spot and forward purchases for delivery on the usage date play an important role in matching the supply and the uncertain demand of energy, because storage capacity for energy, such as electricity, natural gas, and oil, is limited. Transaction costs tend to be larger in spot than forward energy markets near maturity. Partially procuring supply in the forward market, rather than entirely in the spot market, is thus a potentially valuable real option, which we call the forward procurement option. We investigate the optimal value and management of this real option, as well as their sensitivities to parameters of interest. Our research quantifies the value of the forward procurement option on realistic natural gas instances, also suggesting that procuring the demand forecast in the forward market is nearly optimal. This policy greatly simplifies the management of this real option without an appreciable loss of value. We provide some theoretical support for this numerical finding. Beyond energy, our research has potential relevance for the procurement of other commodities, such as metals and agricultural products.

Keywords: Correlated price and demand uncertainty, energy and commodities, newsvendor model, OM-Finance interface, procurement, real options, spot and forward markets, transaction costs, valuation.

1. Introduction

Energy, such as electricity, natural gas, and oil, plays a key economic role in supply chains, as it constitutes a primary input to most industrial and commercial activities (Geman 2005). Spot and forward markets trade energy for immediate and future delivery, respectively. Storage capacity for energy is limited. For example, in 2009 the U.S. natural gas usable storage capacity was about 20% of annual demand (based on data from the Energy Information Administration; EIA 2011). Thus, procurement of supply in spot and forward markets for delivery on the usage date is important in matching a firm’s uncertain demand for energy. Indeed, this research was motivated by our collaboration with an energy reselling company that operates without access to storage capacity.

Transaction costs in energy, and more generally commodity, markets tend to be higher for spot than forward trades near maturity. Specifically, these costs appear to be inversely related to trading volume, a feature for which there is both theoretical support and empirical evidence (Thompson and Waller 1988, Thompson et al. 1993, Bryant and Haigh 2004, Linnainmaa and
Figure 1: The average trading volume for the first thirty-six maturities of the NYMEX natural gas futures contract during the week of March 8, 2010.

Roșu 2009, Roșu 2009). Trading volume typically is thinner far away from the maturity of a futures (forward) contract and during its delivery period, that is, in the spot market, than for dates closer to this maturity – the reduced trading volume in the spot market reflects the reduced availability of the commodity for discretionary purposes in this market. Average transaction costs thus tend to be roughly “U” shaped in time to maturity (see, e.g., Figure 1 in Bryant and Haigh 2004). To partially illustrate this phenomenon, Figure 1 plots the average trading volume for the first thirty-six maturities of the NYMEX natural gas futures contract during the week of March 8, 2010 (this figure does not include data for spot trades). This figure suggests that trading volume in NYMEX natural gas futures is higher for contracts whose maturities are closer to the trading date. Consistent with these observations, in their study of oil derivatives, Trolle and Schwartz (2009, p. 4434) state that “open interest for futures contracts tends to peak when expiration is a couple of weeks away, after which open interest declines sharply.”

This feature of transaction costs in energy markets gives firms engaged in the procurement of energy the option to partially satisfy their uncertain commodity requirement (demand) in the forward market, rather than entirely in the spot market. We call this real option the *forward procurement option*. The literature on this topic from the perspective of differential transaction costs is scant (see §2).

We formulate and analyze a parsimonious model of energy procurement, as well as a richer
and more realistic version of this model, to study the optimal valuation and management of this real option. Our models generalize the classical newsvendor model (Porteus 2002, §1.2) by including both demand and price uncertainty. We apply the valuation approach of Jouini and Kallal (1995), which extends to the case of transaction costs in security trading the risk neutral valuation approach (Luenberger 1998, Chapter 16, Seppi 2002, Birge 2000, Smith 2005). We use our models to investigate, both structurally and numerically, the optimal forward procurement quantity and the value of the forward procurement option, as well as their sensitivities to parameters of interest.

Our analysis provides insights into the management of an important business process in energy supply chains. In particular, by applying our model to realistic natural gas distribution instances, we quantify the value of the forward procurement option to be in the $0.4-2.4M per month range, which amounts to reducing the cost of procuring only in the spot market by about 0.61%-3.52% per month. Moreover, our research suggests the near optimality of a forward procurement policy based only on demand forecasting, which avoids (i) estimating the transaction costs, (ii) modeling the joint distribution of the spot demand and price, and (iii) optimizing the forward procurement decision. This policy thus substantially simplifies the management of the forward procurement option without a considerable loss of its value. We provide some theoretical justification for this numerical observation. Further, this demand forecast forward procurement policy performs near optimally even when the spot and forward transaction costs have similar magnitudes. We observe this behavior despite the optimal forward procurement quantity being considerably below the demand forecast in this case. We attribute this finding to the initial flatness of the objective function for values of the forward procurement quantity above the optimal quantity.

Our analysis relies on assuming equality between the forward price and the expected spot price in the absence of transaction costs. We investigate numerically the effect of relaxing this assumption on our main insights, when arbitrage opportunities from trading in the forward and spot markets are excluded in the presence of transaction costs.

Our insights should be relevant to energy resellers, local distribution companies, and industrial and commercial users, such as food processors, metal and chemical manufacturers, and large restaurant and hotel chains, which purchase large amounts of energy. Potentially, our insights also have broader applicability for the procurement of other commodities, such as metals and agricultural products.

We proceed by reviewing the related literature in §2. We present our base model in §3. We
conduct our structural and numerical analyses of this model in §4 and §5, respectively. We conclude in §6. Online Appendix A summarizes the notation used in the main text. All the relevant proofs of the results stated in the main text are in Online Appendix B. In Online Appendix C we perform a numerical study with a variant of our base model that relaxes the assumption that the forward price and the expected spot price are equal when there are no transaction costs, but precludes arbitrage opportunities from trading in the forward and spot markets when the transaction costs are positive. We formulate and analyze our extended model in Online Appendix D.

2. Literature Review

Our work is related to the operations management literature on long- and short-term contracting in business-to-business settings (Kleindorfer and Wu 2003, Kleindorfer 2008) and the real option literature on energy and commodity applications (Dixit and Pindyck 1994, Sick 1995, Trigeorgis 1996, Smith and McCardle 1999, Seppi 2002). We add to these literatures by defining the forward procurement option and investigating its optimal exercise and valuation.

Real option models of energy and commodity procurement contracts, such as the swing option and other contracts with varying amounts of sourcing flexibility (Li and Kouvelis 1999, Jaillet et al. 2004), address the uncertainty in the purchase price evolution, but neglect demand uncertainty. In contrast, our models capture the joint uncertainty in the spot demand and price. Our research provides novel insights into the effect of demand uncertainty on the optimal forward procurement quantity and the forward procurement option value.

The operations management literature includes various models of the uncertain evolution of the demand forecast in procurement (see, e.g., Hausman 1969, Hausman and Peterson 1972, Graves et al. 1986, Heath and Jackson 1994). We integrate a model of demand uncertainty consistent with these demand forecast evolution models and a model of the uncertain evolution of the spot purchase price in a newsvendor setting (Porteus 2002, §1.2), thus developing and analyzing a novel variant of the newsvendor model.

Various authors have considered the procurement of a commodity in forward and/or spot markets, including Kalymon (1971), Ritchken and Tapiero (1986), Williams (1986, p. 146), Williams (1987), Gurnani and Tang (1999), Gavirneni (2004), Seifert et al. (2004), Berling and Rosling (2005), Gaur and Seshadri (2005), Gaur et al. (2007), Goel and Gutierrez (2009), Nascimento and Powell (2009), Oum and Oren (2010), Berling and Martínez-de-Albéniz (2011), and Boyabatli et al. (2011). Different from these authors, we analyze the impact of the correlation
between the spot demand and price on the optimal forward procurement quantity and the value of the forward procurement option, and we bring to light the near optimality of the demand forecast forward procurement policy.

Ritchken and Tapiero (1986), Seifert et al. (2004), and Oum and Oren (2010) maximize the expected utility of a risk averse procurement manager, while Gaur et al. (2007) and Goel and Gutierrez (2009) model the risk aversion of economic agents via the risk neutral valuation approach. Gaur and Seshadri (2005) consider both approaches. Risk neutral valuation does not admit the presence of transaction costs in security trading. In particular, the model of Goel and Gutierrez (2009) includes differential spot and forward transportation costs, while the transaction costs in our models originate from bid-ask spreads, rather than transportation costs. Thus, while we also consider risk averse economic agents, this difference requires us to adopt a valuation framework that admits transaction costs in security trading, specifically the one of Jouini and Kallal (1995). Applying this valuation framework involves some discussion of the choice of an equivalent probability measure.

3. Base Model

There is a finite horizon that starts at time 0 and ends at time \( T \). The firm satisfies an energy requirement at time \( T \). We refer to this requirement as the spot demand, denoted by \( d \). The spot demand is a random variable as of any time before time \( T \), and becomes known at this time. In our base model, the time \( T \) may correspond to a single date, e.g., a day, or an aggregation of several dates, e.g., a month. Thus, the spot demand is a requirement at a single date, e.g., a daily requirement, in the former case, and an aggregate requirement, e.g., a monthly requirement, in the latter case. Our extended model, discussed in Online Appendix D, refines the aggregate case by splitting the aggregate requirement over several dates.

The firm does not hold any inventory, due to the lack of access to storage facilities, and can satisfy this requirement by procuring energy in the spot market at time \( T \). The firm can also procure supply in advance at time 0 by entering into a forward contract with physical delivery at time \( T \). Time 0 is to be understood as a time near time \( T \) when, as pointed out in §1, the forward transaction costs, introduced below and further discussed in §4.1, are at their lowest level, e.g., two weeks away from time \( T \) in our numerical study presented in §5. In practice, some energy forward contracts can be negotiated to entail delivery on a single date, e.g., a specific day of the month (Belak 2011). When time \( T \) is a single date, our assumption of delivery at time \( T \) of the supply procured in the forward market is realistic for such forward contracts. However,
most energy forward contracts entail delivery at a constant rate during a given period, e.g., a month. When time $T$ is interpreted as an aggregation of multiple dates, in our base model we do not distinguish when the supply procured in the forward market is actually delivered during the delivery period. In contrast, our extended model, presented in Online Appendix D, captures the constant and rateable delivery of typical energy forward contracts.

We denote by $F(0)$ the time 0 nominal price of a forward contract with time $T$ delivery, and simplify it to $F$ (we refer to a price in the absence of transaction costs as a nominal price). This price evolves during the time interval $[0, T]$ as a known stochastic process $F(t) \in \mathbb{R}_+$, $t \in [0, T]$. We focus on the time $T$ forward nominal price, which is the spot nominal price $f$; that is, $F(T) \equiv f$.

Trading in spot and forward markets incurs transaction costs, that is, bid-ask spreads. Consistent with models studied in the finance literature (see, e.g., Constantinides et al. 2007), we model these costs as proportional. Let $A$ and $B \in (0, 1)$. If the firm purchases spot one unit of energy at time $T$, it pays the spot ask price $(1 + A)f$; if the firm sells spot one unit of energy at this time, it receives the spot bid price $(1 - A)f$. At time 0 the firm can forward purchase one unit of energy at the forward ask price $(1 + B)F$. We do not allow the firm to short sell energy forward, as this is suboptimal in the no arbitrage valuation framework that we use, which is explained in §4.1. However, a forward sale at time 0 of one unit of energy can be made at price $(1 - B)F$ (this is useful for the discussion in §4.1). Consistent with theoretical work and empirical evidence on the structure of transaction costs in energy and commodity markets (see §1), the spot transaction costs are larger than the forward transaction costs: $A > B$. That is, we model the observed behavior of transaction costs on average (also recall that we assume that time 0 is near time $T$).

The firm’s time 0 forecast for its time $T$ demand is $D(0)$, which we simplify to $D$. As the forward price, this forecast may evolve as a stochastic process, denoted by $D(t) \in \mathbb{R}_+$, $t \in [0, T]$, which is correlated with the forward nominal price stochastic process. We focus on the spot demand $d$ at time $T$; that is, $D(T) \equiv d$. Since the demand forecast and forward nominal price stochastic processes are correlated, so are the spot demand and nominal price.

For notational convenience, we define $E_t[\cdot] := \mathbb{E}[^{\cdot}\big| D(t), F(t)]$, $\forall (t, D(t), F(t)) \in [0, T] \times \mathbb{R}_+^2$, expectation given $D(t)$ and $F(t)$ with respect to a probability distribution discussed in §4.1. Notice that $E_t[\cdot]$ is not a random variable at time $t$ because $D(t)$ and $F(t)$ are known at this time. That is, our notation does not distinguish between random variables and their realizations; which is which should be clear from the context. We simplify $E_0$ to $\mathbb{E}$. 
The firm needs to decide how much supply $q$ to procure in the forward market at time 0. Such an optimal procurement decision can be obtained by solving the following optimization problem:

$$V := \max_{q \geq 0} \mathbb{E}[(1 - A)f(q - d)^+ - (1 + A)f(q - d)^-] - (1 + B)Fq,$$

(1)

where $(\cdot)^+ := \max\{\cdot, 0\}$ and $(\cdot)^- := -\min\{\cdot, 0\}$. The first and second terms inside the expectation in (1) are the revenue collected from selling excess contracted supply on the spot market and the cost of any supply shortfall, respectively. The third term in (1) is the forward procurement cost. The objective function in (1) and its optimal value $V$ are both expressed in time $T$ money, as money is exchanged only at this time.

Due to the presence of the spot market, the firm always achieves 100% service level. Indeed, $q = 0$ is a feasible solution to (1). This solution corresponds to simply waiting until time $T$, observing the realized demand, and procuring this amount on the spot market. We denote the value of this spot procurement policy as $V^S := -\mathbb{E}[(1 + A)fd]$. We define the value of the forward procurement option, $V^P$, as the additional value obtained by optimally procuring both in the forward and spot markets rather than only in the spot market:

$$V^P := V - V^S.$$

(2)

In other words, $V^P$ is the value of the cost savings accrued by optimally trading both in the spot and the forward markets rather than only in the spot market.

4. Structural Analysis of the Base Model

In this section we analyze the exercise and valuation of the forward procurement option by imposing more structure on our base model; in particular, the demand forecast and forward nominal price processes $D(\cdot)$ and $F(\cdot)$, and, hence, the joint probability distribution of the spot demand random variable $d$ and the spot nominal price random variable $f$. We discuss in §4.1 the valuation framework that we use. We provide a general result in §4.2 and specific results in §4.3 assuming that the spot demand and nominal price are jointly lognormally distributed and satisfy a martingale condition: their expected values, with expectation taken with respect to an equivalent measure in the valuation framework of Jouini and Kallal (1995), are the demand forecast and the forward nominal price, respectively, at the forward trading time. We label this model the martingale lognormal (MLN) model.
4.1 Valuation Framework

When trading futures contracts for a commodity commands transaction costs, as we assume, standard risk neutral valuation (Luenberger 1998, Chapter 16, Seppi 2002) of such a commodity cash flows does not apply. We thus apply the valuation framework with transaction costs of Jouini and Kallal (1995) to value these cash flows. These authors show that the absence of arbitrage in securities markets in the presence of transaction costs is equivalent to the existence of at least one probability measure that converts some process between the bid and the ask price processes of traded securities into a martingale. Any such measure, which Jouini and Kallal (1995) refer to as an equivalent measure or, alternatively, a martingale measure, can be used for valuation purposes in a manner analogous to risk neutral valuation.

To apply this theory here, suppose that the forward transaction costs do not decrease in the time interval $[0, T]$. That is, let $B(\cdot)$ be a time dependent function such that $B(0) \equiv B$, $B(t) \leq B(t')$, $\forall t, t' \in [0, T]$, $t \leq t'$, and $B(T) \equiv A$. Refer to the process $F(t)$, $t \in [0, T]$, as the process for the nominal price of a futures contract for time $T$ delivery; that is, when the transaction costs are absent (here we assume equivalence between futures and forward nominal prices; this condition holds when the risk free interest rate is deterministic, Cox et al. 1981, which we also assume to be the case here). Then, a probability measure under which the process $F(\cdot)$ is a martingale is an equivalent measure in the valuation framework of Jouini and Kallal (1995). That is, under this measure it holds that $F(t) = \mathbb{E}_t[F(t')]$, $\forall t, t' \in [0, T]$, $t \leq t'$, and $F(t) \in \mathbb{R}_+$.

We use this martingale measure to determine the equivalent, according to Jouini and Kallal (1995), probability distribution of the spot nominal price random variable. This is a natural choice of equivalent measure. For example, in the absence of transaction costs, it is the only choice consistent with risk neutral valuation. That is, this equivalent measure is a risk adjusted measure for the futures nominal price process. Under our chosen equivalent measure, the forward nominal price $F$ is thus equal to the expected spot nominal price $f$ (Shreve 2004, p. 244):

$$F = \mathbb{E}[f]. \quad (3)$$

In Online Appendix C we conduct a numerical analysis with a variant of our model that relaxes condition (3).

Condition (3) and $F \in \mathbb{R}_+$ impose the restriction $\mathbb{E}[f] \in \mathbb{R}_+$ for all $(D, F) \in \mathbb{R}_+^2$. To avoid trivial cases, we make the following assumption: $\mathbb{E}[d]$ and $\mathbb{E}[fd] \in \mathbb{R}_+$ for all $(D, F) \in \mathbb{R}_+^2$. 
We assume that changes in the demand forecast process $D(\cdot)$ are uncorrelated with changes in the price of the market portfolio. Hence, we do not risk adjust the evolution of this stochastic process (Smith 2005). (This does not imply that the demand forecast and the forward nominal price processes are uncorrelated.) This assumption is realistic for commercial and residential energy demand, as it is largely determined by the weather (see the related discussion in Hull 2012, Chapter 33).

We focus on the uncertainty in the spot demand and nominal price given the information available at the beginning of the time horizon, that is, $D$ and $F$. All probabilistic statements related to these random variables are under their joint distribution as determined by the previous assumptions. For simplicity, we assume that spot demand is a continuous random variable.

### 4.2 A General Result

Proposition 1 characterizes the optimality condition for problem (1) without assuming a specific (equivalent) joint probability distribution for the spot demand and nominal price. We let $\mathbb{P}$ denote probability.

**Proposition 1** (Optimality condition). Assume that $F > 0$. Consider a positive feasible solution to problem (1). Such a solution is optimal if and only if it satisfies the following condition:

$$E\left[\frac{f}{F}\mathbb{P}\{d \leq q \mid f\}\right] = \frac{1}{2} \left(1 - \frac{B}{A}\right).$$

(4)

If no positive solution to problem (1) satisfies this condition, then zero is the optimal solution to this problem.

As shown in the proof of Proposition 1, the objective function of problem (1) can be expressed as

$$(A - B)Fq - 2A\mathbb{E}[f(q - d)^+] - \mathbb{E}[(1 + Af) F].$$

(5)

The third term in (5) is the value of procuring only in the spot market. The first term in (5) is the value saved by the firm by purchasing an amount of supply $q$ at time 0, irrespective of whether this amount is needed to satisfy the demand at time $T$. The second term in (5) is the reduction in this value if $q$ exceeds this demand. Expression (5) thus shows that problem (1) can be thought of as maximizing the value of the net savings from operating both in the forward and spot markets relative to only operating in the spot market. In other words, the central concern in problem (1) is balancing the savings from procuring in the forward market, rather than only in the spot market, against the cost of buying too much in advance.
To obtain more insight into this trade-off, it is useful to derive condition (4) using marginal analysis. If a positive quantity is procured at time 0, then its associated marginal underage and overage costs are \((1 + A)f - (1 + B)F\) and \((1 + B)F - (1 - A)f\), respectively. Intuitively, such a decision is optimal, in which case we denote it by \(q^*\), if and only if it balances its expected marginal underage and overage costs. Let 1\{\cdot\} be the indicator function of event \{\cdot\}. Exploiting condition (3), the optimal expected marginal underage cost is

\[
E[((1 + A)f - (1 + B)F)1\{d > q^*\}] = E[((1 + A)f - (1 + B)F)(1 - 1\{d \leq q^*\})]
= (A - B)F - E[((1 + A)f - (1 + B)F)1\{d \leq q^*\}].
\]

The optimal expected marginal overage cost is \(E[((1 + B)F - (1 - A)f)1\{d \leq q^*\}]\). We obtain (4) by equating the expressions for these costs, rearranging, and noticing that \(E[f1\{d \leq q^*\}] = E[fE[d \leq q^*|f]] = E[fP\{d \leq q^*|f\}]\).

We can further interpret condition (4) as a generalization of the classical newsvendor critical fractile optimality condition (Porteus 2002, §1.2), which relates the overage probability and the critical ratio. Consider the right hand side of (4). The expectations of the marginal underage and overage costs are \((A - B)F\) and \((A + B)F\), respectively. Thus, the term \((1 - B/A)/2 \in (0, 1/2)\) is the ratio of the expected marginal underage cost and the sum of the expected marginal underage and overage costs, and can be interpreted as a critical ratio. The term \(f/F\), with \(F > 0\), is nonnegative and such that \(E[f/F] = 1\), so that it acts as a weighting random variable for the term \(P\{d \leq q^* | f\}\). Hence, the left hand side of (4) assumes the interpretation of weighted overage probability. Condition (4) relates the weighted overage probability and the critical ratio at optimality. Given this interpretation, that the right hand side of (4) is positive and less than 1/2 emphasizes that the key concern in problem (1) is managing the cost component associated with purchasing too much supply in advance.

### 4.3 Results with the MLN Model

In general, condition (4) can be used as the basis for developing numerical methods to compute an optimal forward procurement decision at time 0 and, hence, the value of the optimal procurement policy. By making specific assumptions on the joint probability distribution of the spot demand and nominal price, we can obtain closed form expressions for these quantities.

We proceed by assuming a lognormal model with unbiased demand forecast. Specifically, we assume that the random variables spot demand \(d\) and spot nominal price \(f\) are jointly lognormally distributed, and \(D = E[d]\). That is, we assume that the natural logarithms of the
Table 1: The time variance, covariance, and riskiness terms.

<table>
<thead>
<tr>
<th>Term</th>
<th>Notation</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>Covariance</td>
<td>$K^C$</td>
<td>$\exp(c s_d s_f)$</td>
</tr>
<tr>
<td>Demand Variance</td>
<td>$K^V$</td>
<td>$\exp(-s_d^2/2)$</td>
</tr>
<tr>
<td>Demand Riskiness</td>
<td>$K^R$</td>
<td>$\exp(z s_d)$</td>
</tr>
</tbody>
</table>

random variables $d$ and $f$ are jointly normally distributed with standard deviations $s_d$ and $s_f$ and means $\ln D - s_d^2/2$ and $\ln F - s_f^2/2$, respectively, and correlation coefficient $c$. This is our MLN model. It satisfies condition (3).

The parameters $s_d$ and $s_f$ are related to the standard deviations $D \sqrt{\exp(s_d^2) - 1}$ and $F \sqrt{\exp(s_f^2) - 1}$ of the spot demand and the spot nominal price, respectively, and, hence, the coefficients of variation (CVs) $\sqrt{\exp(s_d^2) - 1}$ and $\sqrt{\exp(s_f^2) - 1}$, respectively, of these random variables. We thus interpret changes in the values of these parameters as analogous directional changes in these standard deviations and variabilities (CVs). The parameter $c$ is related to the correlation between the spot demand and nominal price, which is

$$e^{c s_d s_f} - 1 \over \sqrt{(e^{s_d^2} - 1)(e^{s_f^2} - 1)}.$$  

Changing the value of $c$ can thus be interpreted as changing (6) in the same direction.

The MLN model is consistent with a version of the single product multiplicative martingale model of demand forecast evolution of Heath and Jackson (1994), which in our case essentially reduces to the model of Hausman (1969). It is also consistent with the futures price evolution model of Black (1976), as well as the one implied by assuming that the spot nominal price follows a mean reverting process (see, e.g., Jaillet et al. 2004).

The optimal forward procurement quantity. Proposition 2 establishes a closed form expression for the optimal forward procurement quantity under the MLN model. This proposition uses the covariance, demand variance, and demand riskiness terms defined in Table 1. The covariance term depends on the covariance between the random variables $\ln d$ and $\ln f$; that is, $c s_d s_f$. The demand variance term depends on the variance of the random variable $\ln d$; that is, $s_d^2$. We denote by $z$ the $(1 - B/A)/2$-th percentile of the standard normal distribution. The demand riskiness term depends on the product of $z$, which is negative as the critical ratio $(1 - B/A)/2$ is positive and less than 1/2, and the standard deviation of $\ln d$ (the riskiness label is motivated by the negative sign of the exponent of this term and contrasts the safety label that would be appropriate if this sign were positive).
Table 2: Comparative statics of the optimal forward procurement quantity under the MLN model; - , 0, and + denote a negative, null, and positive effect, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Sign of the Effect</th>
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<tbody>
<tr>
<td>$D$</td>
<td>+</td>
</tr>
<tr>
<td>$c$</td>
<td>+</td>
</tr>
<tr>
<td>$s_f$</td>
<td>$c &lt; 0$: -; $c = 0$: 0; $c &gt; 0$: +</td>
</tr>
<tr>
<td>$s_d$</td>
<td>$c \leq 0$: -; $c &gt; 0$: + for $s_d \in (0, \bar{s}_d^f)$, - for $s_d \in (\bar{s}_d^f, \infty)$</td>
</tr>
<tr>
<td>$B/A$</td>
<td>-</td>
</tr>
</tbody>
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Proposition 2 (Optimal forward procurement quantity with the MLN model). Under the MLN model the optimal amount of supply to procure forward in problem (1) is

$$q^* = K^C K^V K^R D.$$  

(7)

This proposition shows that under the MLN model the optimal quantity to procure forward is the current demand forecast $D$ scaled by the product of the covariance, demand variance, and demand riskiness terms. In isolation, the demand variance and riskiness terms make the optimal amount of supply to procure in the forward market decrease below the demand forecast. The effect of the covariance term depends on the sign of the correlation coefficient $c$. By itself, this effect is to make $q^*$ decrease below $D$ if $c < 0$ and increase above $D$ if $c > 0$. It is easy to verify that $q^*$ is at least $D$ if and only if $c \geq (s_d/2 - z)/s_f > 0$. A sufficiently large positive value of $c$ is thus needed for the optimal procurement quantity to exceed $D$.

Comparative statics of the optimal forward procurement quantity. Under the MLN model the optimal forward procurement quantity does not depend on the forward price. It instead depends on the standard deviation of the natural logarithm (log) of the spot nominal price and the correlation coefficient between the log spot demand and nominal price. Corollary 1, which follows easily from Proposition 2, establishes the comparative statics of the optimal forward procurement quantity with respect to other quantities of interest. We define $\bar{s}_d^f := (z + cs_f)^+$. Table 2 summarizes these comparative statics.

Corollary 1 (Comparative statics of $q^*$ with the MLN model). Consider the MLN model. Ceteris paribus, the optimal forward procurement quantity (1) increases in $D$; (2) increases in $c$; (3) decreases in $s_f$ if $c < 0$, does not depend on $s_f$ if $c = 0$, and increases in $s_f$ if $c > 0$; (4) decreases in $s_d$ if $c \leq 0$, and increases in $s_d$ when $s_d$ increases up to $\bar{s}_d^f$ and decreases in $s_d$ for values of this parameter that exceed $\bar{s}_d^f$ if $c > 0$; and (5) decreases in $B/A$. 

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The increase of the optimal forward procurement quantity in the demand forecast, \(D\), is obvious as this amounts to an increase in the expected spot demand.

The behavior of this optimal quantity in the correlation coefficient, \(c\), can be explained by noticing that increasing the value of \(c\) decreases the weighted overage probability. Intuitively, procuring in the forward market thus becomes more appealing.

The effect of changing the log spot nominal price standard deviation, \(s_f\), depends on the sign of the correlation coefficient \(c\) and modulates its effect. An increase in \(s_f\) weakens the effect of \(c\) if \(c < 0\), strengthens it if \(c > 0\), and does not affect it if \(c = 0\).

Changing the log spot demand standard deviation, \(s_d\), affects the demand variance, demand riskiness, and covariance terms. If \(c \leq 0\), an increase in \(s_d\) affects all of these terms in the same direction, so that the optimal forward procurement quantity decreases. If \(c > 0\), the increase in the covariance term due to an increase in \(s_d\) dominates the corresponding decreases in the demand variance and riskiness terms, and the optimal forward procurement quantity also increases; but when \(s_d\) grows sufficiently large, the decreases in the latter two terms dominate the increase in the covariance term, and the optimal forward procurement quantity also decreases.

The decrease of the optimal forward procurement quantity in the transaction cost ratio, \(B/A\), is intuitive.

**Valuations.** Proposition 3 provides closed form expressions for the values of the spot procurement policy, the optimal forward procurement policy, and the forward procurement option, as well as a bound on the relative value of this option. Two of these expressions are closed form in the sense that they depend, through the term \(\alpha := 2A\Phi(z - s_d)\), on the cumulative distribution function of the standard normal distribution, \(\Phi(\cdot)\), which can be readily evaluated in a spreadsheet.

**Proposition 3 (Valuations with the MLN model).** Consider model MLN. The values of the spot procurement policy, the optimal procurement policy, the forward procurement option, and a bound on the relative value of this option, respectively, are

\[
V^S = -(1 + A)K^CFD, \quad (8) \\
V = V^S + \alpha K^C FD, \quad (9) \\
V^P = \alpha K^C FD, \quad (10) \\
\frac{V^P}{|V^S|} < \frac{A}{1 + A}. \quad (11)
\]
Table 3: Comparative statics of the value of the forward procurement option with the MLN model; –, 0, and + denote a negative, null, and positive effect, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Sign of the Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>+</td>
</tr>
<tr>
<td>$F$</td>
<td>+</td>
</tr>
<tr>
<td>$c$</td>
<td>+</td>
</tr>
<tr>
<td>$s_f$</td>
<td>$c &lt; 0$: –; $c = 0$: 0; $c &gt; 0$: +</td>
</tr>
<tr>
<td>$s_d$</td>
<td>either $c \leq 0$ or $c &gt; 0$ and $C$ holds: –</td>
</tr>
<tr>
<td></td>
<td>$c &gt; 0$ and $\overline{C}$ holds: + for $s_d \in (0, \overline{s}_d^V)$, – for $s_d \in (\overline{s}_d^V, \infty)$</td>
</tr>
<tr>
<td>$B/A$</td>
<td>–</td>
</tr>
</tbody>
</table>

The value of the spot procurement policy is the product of the transaction cost term $-(1+A)$, the covariance term, the forward nominal price, and the demand forecast. The value of the optimal procurement policy is the value of the spot procurement policy plus the value of the net savings that accrue to the firm by optimally procuring in the forward market, that is, the term $\alpha K^C FD$ (see the discussion following expression (5)). By definition (2), the value of these net savings is the value of the forward procurement option. Obviously, the value of this option cannot be negative. Moreover, this value relative to the absolute value of the spot procurement option is bounded above by less than half the spot bid-ask spread (this spread is $2A$). That is, this is the most that can be gained from optimally procuring both in the forward and the spot markets, rather than only in the spot market.

**Comparative statics of the value of the forward procurement option.** Corollary 2 establishes some of the comparative statics of the value of the forward procurement option. We denote by $\phi(\cdot)$ the standard normal density function. We abbreviate by $C$ the condition $s_f c \leq \phi(z)/\Phi(z)$, and by $\overline{C}$ its complement. If $\overline{C}$ is true, then we define $\overline{s}_d^V$ as that value of $s_d$ that satisfies the equality $s_f c = (\phi(z - s_d)/\Phi(z - s_d)$ ($\overline{s}_d^V$ exists and is positive as shown in the proof of Corollary 2).

**Corollary 2 (Comparative statics of $V^P$ with the MLN model).** Consider model MLN. Ceteris paribus, the value of the forward procurement option, $V^P$, (1) increases in $D$; (2) increases in $F$; (3) increases in $c$; (4) decreases in $s_f$ if $c < 0$, is not affected by $s_f$ if $c = 0$, and increases in $s_f$ if $c > 0$; (5) decreases in $s_d$ if either $c \leq 0$ or $c > 0$ and $C$ holds, and increases up to the threshold $\overline{s}_d^V$ and decreases thereafter in $s_d$ if $c > 0$ and $\overline{C}$ holds; and (6) decreases in $B/A$.

Consider part (1). From expression (5), the value of the forward procurement option is the difference between the value of the savings and the value of the overprocurement cost when
acting optimally in the forward market. These values are both linear in the forward nominal price, \( F \) (this is obvious for the savings; see (28) in Online Appendix B for the overprocurement cost). The value of these savings is obviously greater than the value of this overprocurement cost for every given such price. The difference between the former and the latter values thus increases in this price. A similar argument holds for part (2) with respect to the spot demand forecast, \( D \). Parts (3)-(4) mirror how the correlation coefficient and the log spot nominal price standard deviation affect the optimal forward procurement quantity. Part (5) is related to the behavior of the optimal forward procurement quantity in the log spot demand standard deviation. Part (6) is intuitive.

**The demand forecast procurement policy.** Determining the optimal forward procurement quantity requires knowledge of the parameters of the joint spot demand and nominal price distribution, as well as the spot and forward market transaction costs. It is thus of interest to consider a forward procurement policy that does not require this information and investigate the suboptimality of such a policy.

We take this policy to be the one that procures forward an amount equal to the demand forecast, that is, \( q = D \). We choose this policy as it does not require knowledge of the transaction costs and ignores all but one of the parameters of the MLN model; that is, \( D \).

We denote the value of the demand forecast procurement policy by \( V^D \). It follows from (5) and Lemma 2 in Online Appendix B that this value with the MLN model is

\[
V^D = V^S + \left\{ A - B - 2A \left[ \Phi \left( \frac{s_d}{2} - cs_f \right) - K^C \Phi \left( -\left( \frac{s_d}{2} + cs_f \right) \right) \right] \right\} FD. \tag{12}
\]

The demand forecast procurement policy is optimal when there is no demand uncertainty (irrespective of the specific spot nominal price distribution used). This is because the choice in this case is between procuring an amount \( D \) in the forward market if \((1 + B)F \leq (1 + A)\mathbb{E}[f] = (1 + A)F \) and procuring this amount in the spot market otherwise, and \( B < A \).

Under the MLN model, this policy is optimal when the log spot demand standard deviation is equal to \( 2\sigma_d^q > 0 \), because in this case \( q^* = D \). Proposition 4 establishes additional conditions such that the demand forecast policy outperforms the spot procurement policy, and is hence able to capture a fraction of the value of the forward procurement option, and bounds both the absolute and relative suboptimality of the demand forecast procurement policy in a specific case.
Proposition 4 (Performance of the demand forecast procurement policy with the MLN model). Consider the MLN model. (1) There exists a value \( \bar{s}_d^D > 2\bar{s}_d^q \) that depends on the problem parameters such that \( V^D > V^S \) when and only when \( s_d \in (0, \bar{s}_d^D) \) and \( V^D = V^S \) when \( s_d = \bar{s}_d^D \). (2) If \( s_d \in (2\bar{s}_d^q, 2cs_f] \) then it holds that

\[
\begin{align*}
V - V^D &< FDB, \\
\frac{V - V^D}{|V|} &< \frac{B}{Kc}.
\end{align*}
\]

Part (1) of Proposition 4 indicates that the demand forecast policy outperforms the spot procurement policy when the variability in the spot demand is “not too large.” The bounds (13) and (14) on the absolute and relative suboptimality of the demand forecast procurement policy require the log spot demand and nominal price to be positively correlated, which is realistic. These bounds can be small because realistic values of the forward transaction cost \( B \) can be small (see §5.1). In particular, the bound (13) is less than the suboptimality of the spot procurement policy, that is, the value of the forward procurement option, if the forward transaction cost is sufficiently small, that is, \( B < 2A\Phi(z - s_d) \exp(cs_ds_f) \). There are realistic parameter values that satisfy this condition, as discussed in §5.2. Overall, Proposition 4 suggests that the demand forecast policy can capture a substantial amount of the value of the forward procurement option in realistic cases.

It is possible to establish bounds on the suboptimality of the forward procurement policy in more cases than the one specified in part (2) of Proposition 4. It is however challenging to make these bounds as informative as the ones stated in this proposition.

5. Numerical Study with the Base Model

In this section we report the results of a numerical study conducted with our base model to assess the value of the forward procurement option and the performance of the demand forecast procurement policy, as well as illustrate some of the comparative statics of this option and its optimal exercise. We present the instances used as test beds in §5.1. We discuss our results in §5.2.

5.1 Instances

In our study, the firm is a hypothetical natural gas distributor operating in the U.S. We let the length of the time horizon be two weeks, and thus set \( T \) equal to \( 14/365 \). The end of the time horizon corresponds to February 2011; that is, we use the aggregate interpretation of our base
We take the firm’s February 2011 demand forecast as of time 0 to be 14,593,766MMBtu, which corresponds to the February 2001 demand faced by the utility considered in the study of Muthuraman et al. (2008, Table 1, p. 1143). Here, we make the convenient assumption that this figure is indicative of the demand forecast of this firm for February 2011. We use $4.4315/MMBtu as the forward nominal price at the beginning of the time horizon; this is the average of the low and high prices of the NYMEX natural gas futures contract for delivery in February 2011 observed on January 14, 2011.

We specify the parameters of the MLN model by employing the single product multiplicative martingale model of demand forecast evolution of Heath and Jackson (1994) and the seasonal mean reverting model for energy spot prices discussed by Jaillet et al. (2004). Specifically, the demand forecast for time $T$ evolves as a geometric Brownian motion with volatility $\sigma_D$ and zero drift, so that each demand forecast is unbiased; that is, the expected value of the spot demand is equal to the current demand forecast. The log deseasonalized natural gas spot nominal price is $\chi(t) := \ln(f(t)/S(t))$, where $f(t)$ is the spot nominal price at time $t$ and $S(t)$ is the deterministic seasonality factor for this price. This log price evolves as a mean reverting process with speed of mean reversion $\kappa$, risk adjusted mean reversion level $\xi$, and volatility $\sigma_\chi$. The dynamics of these processes in the time interval $[0, T]$ are

\begin{align}
    dD(t) &= \sigma_D D(t) dZ_D(t), \tag{15} \\
    d\chi(t) &= \kappa[\xi - \chi(t)] dt + \sigma_\chi dZ_{\chi}(t), \tag{16} \\
    dZ_{D}(t)dZ_{\chi}(t) &= \rho dt, \tag{17}
\end{align}

where $dZ_D(t)$ and $dZ_\chi(t)$ are increments to standard Brownian motions with instantaneous correlation coefficient $\rho$, and $dt$ is an infinitesimal time increment. Under model (15)-(17), the parameters of the MLN model are as follows:

\begin{align*}
    s_d &= \sigma_D \sqrt{T}, \\
    s_f &= \sigma_\chi \sqrt{\frac{1 - e^{-2\kappa T}}{2\kappa}}, \\
    c &= \frac{\rho(1 - e^{-\kappa T})/\kappa}{\sqrt{T(1 - e^{-2\kappa T})/(2\kappa)}}.
\end{align*}

Given that $F$ is known, the values of the parameters $\chi(0)$, $\xi$, and $S(T)$ are not needed, because they are assumed to satisfy condition (3), which reduces to

\[ F = S(T) \exp \left( \chi(0)e^{-\kappa T} + (1 - e^{-\kappa T})\xi + \frac{\sigma_\chi^2}{4\kappa}(1 - e^{-2\kappa T}) \right). \]
However, the values of these parameters are needed when dealing with our extended model, as discussed in Online Appendix D.3.

We set a base value for the log demand forecast volatility, $\sigma_D$, to make the CV of spot demand equal to 0.05. We have confirmed with practitioners that this is a reasonable figure for natural gas demand. In our specification of the MLN model, the CV of spot demand given the information available at time 0 is $\sqrt{\exp(\sigma_D^2 T) - 1}$. Thus, for given values of $T$ and this CV, the corresponding value of $\sigma_D$ is $\sqrt{\ln(1 + CV^2)/T}$. For $T$ equal to 14/365 and the CV of spot demand equal to 0.05, we obtain a value for $\sigma_D$ equal to 0.2551. To obtain insights into the effect of this parameter on the value of the forward procurement option, we also consider values for $\sigma_D$ equal to 0.1276, 0.3824, 0.5093, 0.6358, 0.7616, 0.8868, 1.0112, 1.1347, 1.2572, 1.3786, and 1.4989, which roughly correspond to spot demand CV values in the 0.025-0.3 range in increments of 0.025, respectively. Moreover, our range of values for $\sigma_D$ includes the 0.6458-1.1157 range of electricity log demand volatilities reported in Pilipovic (2007, Figure 11-28, p. 343). Although we deal with natural gas, rather than electricity, demand, the overlap between these ranges provides some support for our choice of values of the parameter $\sigma_D$.

For the log deseasonalized spot nominal price speed of mean reversion and volatility, $\kappa$ and $\sigma_X$, we consider base values of 1.0547 and 0.6696, which are the estimates of these parameters obtained by Lai et al. (2011, Table 2) using NYMEX data. Similar to $\sigma_D$, we consider additional values for $\sigma_X$, namely 0.2696, 0.3696, 0.4696, 0.5696, and 0.7696, leaving the value of the parameter $\kappa$ at its base level.

We set a base value for the instantaneous correlation coefficient, $\rho$, by making the correlation between the spot demand and nominal price with $T = 14/365$ equal to 0.2 (this value is used in the study of Seifert et al. 2004). Specifically, in our specification of the MLN model the correlation between the spot demand and nominal price is $\{\exp(\rho \sigma_D \sigma_X [1 - \exp(-\kappa T)]/\kappa) - 1\} / (G_D G_f)$, with $G_D := \sqrt{\exp(\sigma_D^2 T) - 1}$ and $G_f := \sqrt{\exp(\sigma_X^2 [1 - \exp(-2\kappa T)]/(2\kappa)) - 1}$. Thus, for given values of this correlation, denoted by CORR, and $T$, the corresponding value of $\rho$ is $\ln(1 + G_D G_f \text{CORR}) / \{\sigma_D \sigma_X [1 - \exp(-\kappa T)]/\kappa\}$. For $T = 14/365$, $\sigma_D = 0.26$, and $\sigma_X = 0.6696$, we obtain $\rho = 0.1968$. Based on the argument made by Seifert et al. (2004) that the correlation between a commodity price and the demand for this commodity faced by a firm should be positive, we also consider values for $\rho$ equal to 0.0246, 0.0492, 0.0738, 0.0984, 0.1230, 0.1476, 0.1722, 0.2952, 0.3934, 0.4916, 0.5897, and 0.6878, each of which roughly corresponds to an equal CORR value. We have verified with practitioners that this range includes realistic values. In addition, this range includes the 0.0267-0.2134 range of correlations between electricity
Table 4: Parameter values used in our numerical study with our base model; when there are multiple values for a parameter, the boldface value corresponds to the base case.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$14/365$</td>
</tr>
<tr>
<td>$D$</td>
<td>$14,593,766\text{MMBtu}$</td>
</tr>
<tr>
<td>$F$</td>
<td>$4.475/\text{MMBtu}$</td>
</tr>
<tr>
<td>$\sigma_D$</td>
<td>$0.1276, 0.2551, 0.3824, 0.5093, 0.6358, 0.7616, 0.8868, 1.0112, 1.1347, 1.2572, 1.3786, 1.4989$</td>
</tr>
<tr>
<td>$\sigma_X$</td>
<td>$0.2696, 0.3696, 0.4696, 0.5696, 0.6696, 0.7696$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$1.0547$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$0.0246, 0.0492, 0.0738, 0.0984, 0.1230, 0.1476, 0.1722, 0.1968, 0.25%, 2.5%$</td>
</tr>
<tr>
<td>$A$</td>
<td>$3.75%$</td>
</tr>
<tr>
<td>$B$</td>
<td>$0.025%, 0.25%, 2.5%$</td>
</tr>
</tbody>
</table>

Table 5: Summary of our numerical results with the base model.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Base Case</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^P$</td>
<td>$2,315,354$</td>
<td>$502,975-2,366,699$</td>
</tr>
<tr>
<td>$V^P/</td>
<td>V^S</td>
<td>$</td>
</tr>
<tr>
<td>$q^*/D$</td>
<td>99.96%</td>
<td>72.13-100.43%</td>
</tr>
<tr>
<td>$V - V^D$</td>
<td>$3$</td>
<td>$0-259,780$</td>
</tr>
<tr>
<td>$(V - V^D)/</td>
<td>V</td>
<td>$</td>
</tr>
</tbody>
</table>

price and demand reported in Pilipovic (2007, Table 11-2, p. 342). Despite the difference in commodity, the overlap between our and this range gives some support for our choice of values of the parameter $\rho$.

In terms of the transaction costs, we rely on indicative figures provided to us by practitioners at a major U.S. energy trading company and at a major U.S. natural gas broker. These figures are consistent with our assumed transaction cost structure. We set the forward bid-ask spread $(2B)$ equal to 0.05%, and the spot bid-ask spread $(2A)$ to 7.5%. We also consider forward bid-ask spreads equal to 0.5% and 5% to quantify the impact of the forward and spot transaction cost differential on the forward procurement option value and optimal quantity.

Table 4 summarizes the parameters used to generate our 2,808 (= $12 \cdot 6 \cdot 13 \cdot 3$) instances.

5.2 Results

In discussing our results, we focus on the value of the forward procurement option, the optimal forward procurement quantity, some of their comparative statics, and the performance of the demand forecast procurement policy. Table 5 summarizes our results.
The value of the forward procurement option. The upper bound (11) on the relative value of the forward procurement option is 3.61%. In the base case, the relative value of this option is 3.45%, which is 95.57% of this bound. Thus, the bound (11) is informative in this case. Across all our instances, the relative value of the forward procurement option varies in between 0.75% and 3.52%, which correspond to 20.78% and 91.73% of the value of the bound (11). In particular, the ratios of the relative value of the forward procurement option and this bound vary in between 76.45% and 97.51%, 70.64% and 91.41%, and 20.78% and 32.13% when the forward transaction cost is 0.025%, 0.25%, and 2.5%, respectively. The bound (11) thus becomes looser when the forward transaction cost increases. This behavior is expected given that the value of the forward procurement option decreases when the value of $B$ increases, due to the corresponding decrease in the value of $z$, but this bound is unaffected by such changes.

In absolute terms, the forward procurement option is worth $2,315,354 per month in the base case, and ranges from $502,975 to $2,366,699 per month across all of our instances. Since these are reductions in the cost of the spot procurement policy achieved by the optimal forward procurement policy, these findings suggest that a firm may derive substantial benefit from optimally exploiting the differential transaction costs between the forward and spot markets.

Comparative statics of the value of the forward procurement option. Proposition 3 implies that the ratio of the forward procurement option and the absolute value of the spot procurement policy is equal to $\frac{\alpha}{1+A}$, which only depends on the transaction cost parameters, $A$ and $B$, the log demand forecast volatility, $\sigma_D$, and the length of the time horizon, $T$ (as $\alpha$ depends on $\sigma_D$ and $T$, via $s_d = \sigma_D \sqrt{T}$, and $z$, which in turn depends on $A$ and $B$). Increasing the log demand forecast volatility decreases these ratios by less than 1%. Specifically, these ratios vary in between 2.76% and 3.52%, 2.55% and 3.30%, and 0.75% and 1.16% when the forward transaction cost is 0.025%, 0.25%, and 2.5%, respectively. As expected, each set of ratios decreases when the forward transaction cost increases. More interesting, each set of ratios decreases below 2% only when this cost increases by two orders of magnitude, that is, from 0.025% to 2.5%. Further, even when the spot and forward transaction costs share the same order of magnitude, the forward procurement option retains some value.

Even though these ratios do not vary by more than 1%, for a given value of the forward transaction cost, the monetary value of the forward procurement option varies in a more substantial manner. Specifically, it ranges in between $1,850,447 and $2,366,699, $1,712,886 and $2,221,029, and $502,975 and $780,464 per month when the forward transaction cost is 0.025%,
0.25%, and 2.5%, respectively. These variations in the option value are almost completely attributable to changes in the log demand forecast volatility, $\sigma_D$ (increasing $\sigma_D$ decreases the option value in our instances). That is, the value of the forward procurement option is rather insensitive to changes in the log deseasonalized spot nominal price volatility, $\sigma_\chi$, and the instantaneous correlation coefficient, $\rho$, on our instances.

The optimal forward procurement quantity and its comparative statics. The optimal forward procurement quantity, $q^*$, is 99.96% of the demand forecast, $D$, in the base case. It varies in between 72.13% and 100.43% of this forecast across all our instances. More specifically, it ranges in between 95.58% and 100.43%, 93.49% and 100.01%, and 72.13% and 97.83% of the demand forecast when the forward transaction cost is 0.025%, 0.25%, and 2.5%, respectively. The optimal forward procurement quantity thus drops substantially below the demand forecast in some of the instances in which the spot and forward transaction costs are of the same order of magnitude.

To explain this observation, recall from Proposition 2, specifically (7), that the optimal forward procurement quantity is the demand forecast scaled by the product of the demand variance, demand riskiness, and covariance terms. In isolation, the demand variance and riskiness terms make the optimal forward procurement quantity decrease below the demand forecast, while the covariance term has the opposite effect when the instantaneous correlation coefficient is positive, which is the case in our numerical study. Across all our instances, the covariance and demand variance terms range from 1.0000 to 1.0303 and from 0.9578 to 0.9997, respectively, and their product ranges from 0.9582 to 1.0052. Thus, the combined effect of these two terms is to keep the optimal forward procurement quantity close to the demand forecast. In contrast to these two terms, the demand riskiness term depends on the forward transaction cost. This term varies in between 0.9976 and 0.9998, 0.9757 and 0.9979, and 0.7528 and 0.9761, when this cost is 0.025%, 0.25%, and 2.5%, respectively. Consequently, the optimal forward procurement quantity drops considerably below the demand forecast only on some instances when the spot and forward transaction costs share the same order of magnitude.

Similar to the value of the forward procurement option, the optimal forward procurement quantity is more sensitive to changes in the volatility of the log demand forecast than changes in the volatility of the log deseasonalized spot nominal price and their instantaneous correlation coefficient.
The performance of the demand forecast procurement policy. Proposition 4 suggests that the demand forecast procurement policy can perform well in some cases. Specifically, the value of the bound (13) on the absolute suboptimality of the demand forecast procurement policy is less than the value of the forward procurement option on 164 out of 291 instances that satisfy the condition for the bound validity (these 164 instances correspond to all the 38 instances with $B = 0.025\%$ and all the 126 instances with $B = 0.25\%$ for which this bound is valid; that is, the value of this bound exceeds the value of the forward procurement option on all the 127 instances with $B = 2.5\%$ for which this bound is valid). Across these 164 instances, their respective forward transaction costs, that is, 0.025% and 0.25%, turn out to be the largest values of the bound (14) on the relative suboptimality of this policy. More broadly, our numerical analysis of the optimal forward procurement quantity suggests that the demand forecast procurement policy should be near optimal on all the instances for which the spot and forward transaction costs have different orders of magnitude, because on these instances the demand forecast is close to the optimal forward quantity. Indeed, the suboptimality of the demand forecast procurement policy is at most 0.01% and 0.02%, which translate to $6,613$ and $12,564$ per month, when the forward transaction costs are 0.025% and 0.25%, respectively.

It is however unclear if the demand forecast procurement policy would also perform well on the instances for which the forward and spot transaction costs have the same order of magnitude, because on these instances the demand forecast can be substantially above the optimal forward procurement quantity, that is, by as much as $32.87\%$ ($= (1/0.7526 - 1)\%$). Our numerical results reveal that the demand forecast policy is near optimal even on these instances: its suboptimality is at most 0.39% ($259,780$ per month).

This finding indicates that when the spot and forward transaction costs have the same order of magnitude, so that the demand forecast can considerably exceed the optimal forward procurement quantity, the objective function of problem (1) is relatively flat to the right of its optimal solution. To gain some intuition on this behavior, let $\eta$ be a positive number and consider the loss from procuring $\eta q^*$ more than is optimal in the forward market. This loss can be easily shown to be

$$2A\mathbb{E} \left[ f((1 + \eta)q^* - d)^+ - f(q^* - d)^+ \right] - (A - B)Fq^*\eta. \tag{18}$$

Given the discussion following (5) in §4.2, the first term in this difference is the value of the additional reduction in savings from sourcing in the forward market, due to the forward supply exceeding the realized demand, when procuring $(1 + \eta)q^*$ rather than $q^*$; the second term is
the value of the savings obtained from overprocuring in the forward market the amount $q^* \eta$. Because (18) cannot be negative, the second term of this difference never exceeds the first term. Moreover, while both these terms become larger when $\eta$ increases, it is easy to verify that their difference is an increasing and convex function of $\eta$. That is, the first term grows faster than the second term as $\eta$ increases. Our finding indicates that initially the increments in these two terms tend to offset each other, and hence the initial growth in their difference is slow.

These results suggest that achieving near optimal management of the forward procurement option requires only demand forecasting, rather than estimating the transaction costs, the joint distribution of the spot demand and nominal price, and optimizing the forward procurement quantity. Thus, these results suggest that the demand forecast procurement policy is appealing for practical implementation.

6. Conclusions

In this paper we study the optimal valuation and management of the forward procurement option, a real option that arises from the differential transaction costs in spot and forward energy, and more broadly commodity, markets. We do this by formulating, analyzing, and applying to data a parsimonious procurement model with correlated spot demand and nominal price random variables, as well as an extension thereof. In particular, we quantify the value of the forward procurement option on realistic natural gas distribution instances. Our numerical work suggests that procuring the demand forecast in the forward market is near optimal. We provide some theoretical support for this numerical observation. This policy thus substantially simplifies the management of this real option without a considerable loss of its value. We observe that the demand forecast forward procurement policy performs near optimally even when the magnitudes of the spot and forward transaction costs are similar. In this case, the optimal forward procurement quantity is substantially smaller than the demand forecast, but the near optimality of the demand forecast policy is due to the observed initial flatness of the objective function to the right of the optimal forward procurement quantity. Our findings are relevant to energy resellers and local distribution companies, but retain potential significance in other industrial and commercial contexts.

Our work can be extended in several directions. We consider a single procurement date in the forward market. It would be of interest to consider a more dynamic model with additional procurement dates later on in the forward market. This model may yield some additional
benefit relative to our models, but would require estimating the dynamics of the transaction costs in a more granular fashion.

Firms may face limits to the amount of trading that they can perform in the forward market at a given date, e.g., due to limited working capital availability, counterparty risk, as well as supply chain disruptions (Tomlin and Wang 2012) that may affect the availability of contracted supply. Our models could be extended to consider these features.

The transaction costs are deterministic in our models. This simplification captures the average behavior of these costs over time. A more refined approach might model the stochastic evolution of transaction costs, possibly linking it to the one of trading volume, and the relationship between the evolution of these costs and the ones of demand and price. Relevant empirical research would be useful in this respect.

In this paper we use the valuation approach of Jouini and Kallal (1995). Considering the spot and forward procurement choices of multiple firms within an equilibrium model would allow one to assess the limitations of this no-arbitrage approach applied to the valuation and management of the forward procurement option.

Despite the limited availability of energy storage capacity in practice, our models could be extended to include storage in a manner similar to the inventory models reviewed in §2. In particular, it would be of interest to assess the value of storage for energy users in practical settings (see, e.g., Butler and Dyer 1999).

Acknowledgments

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References

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Online Appendix

A. Summary of the Notation Used in the Main Text

Table 6 summarizes the notation used in the main text.

B. Proofs of the Results in the Main Text

Proof of Proposition 1 (Optimality condition). Define \( v(q) := (1 - A)f(q - d)^+ - (1 + A)f(q - d)^- \). Since \((\cdot)^- \equiv (-\cdot)^+\), it holds that \( v(q; d, f) = (1 + A)f(q - d) - 2Af(q - d)^+ \). This equality and condition (3) imply that the objective function of problem (1) can be rearranged as

\[
\mathbb{E}[v(q)] - (1 + B)Fq = \mathbb{E}[(1 + A)f(q - d) - 2Af(q - d)^+] - (1 + B)Fq
\]

\[
= (A - B)Fq - 2A\mathbb{E}[f(q - d)^+] - \mathbb{E}[(1 + A)f].
\]

Thus, solving problem (1) is equivalent to solving

\[
\max_{q \geq 0} (A - B)Fq - 2A\mathbb{E}[f(q - d)^+]. \tag{19}
\]

Recall that condition (3) and \( F \in \mathbb{R}_+ \) restrict \( \mathbb{E}[f] \) to be in \( \mathbb{R}_+ \). It is thus easy to show that the objective function of problem (19) is proper concave in \( q \) (see Rockafellar 1970, p. 24 for the definition of proper concave function), and tends to \(-\infty\) as \( q \) tends to \( \infty \). Hence, a positive solution to this problem is optimal if and only if it satisfies the first order optimality condition

\[
(A - B)F - 2A\mathbb{E}[f1\{d \leq q\}] = 0, \tag{20}
\]

where \( 1\{\cdot\} \) is the indicator function of event \( \{\cdot\} \). The expectation in (20) can be expressed as \( \mathbb{E}[f1\{d \leq q\}] = \mathbb{E}[f\mathbb{E}[1\{d \leq q\}|f]] = \mathbb{E}[f\mathbb{P}\{d \leq q|f\}] \). Using this equality in (20), rearranging, and exploiting the assumed positivity of \( F \) yields

\[
\mathbb{E} \left[ \frac{f}{F}\mathbb{P}\{d \leq q|f\} \right] = \frac{1}{2} \left( 1 - \frac{B}{A} \right).
\]

If no positive solution satisfies this condition, the concavity in \( q \) of the objective function of problem (19) implies that zero is an optimal solution to this problem. □

Lemma 1 (Lognormal partial moment). Denote by \( g(w; \mu, \sigma) \) and \( \phi(y; \mu, \sigma) \) the probability density functions of a lognormal random variable \( W \) and a normal random variable \( Y \), respectively, with parameters \( \mu \) and \( \sigma \), and by \( \Phi(\cdot) \) the cumulative distribution function of the standard
Table 6: Summary of the notation used in the main text.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0, T$</td>
<td>Forward, spot trading times</td>
</tr>
<tr>
<td>$t, t'$</td>
<td>Times in set $[0, T]$</td>
</tr>
<tr>
<td>$F(t), F$</td>
<td>Time $t, 0$ nominal forward prices</td>
</tr>
<tr>
<td>$f(t), f$</td>
<td>Time $t, T$ nominal spot prices</td>
</tr>
<tr>
<td>$D(t), D$</td>
<td>Time $t, 0$ demand forecasts</td>
</tr>
<tr>
<td>$d$</td>
<td>Time $T$ spot demand</td>
</tr>
<tr>
<td>$E_t, E$</td>
<td>Time $t, 0$ conditional expectations</td>
</tr>
<tr>
<td>$\mathbb{P}$</td>
<td>Probability</td>
</tr>
<tr>
<td>$B(t), B$</td>
<td>Times $t, 0$ forward transaction costs</td>
</tr>
<tr>
<td>$A$</td>
<td>Time $T$ spot transaction cost</td>
</tr>
<tr>
<td>$q, q^*$</td>
<td>Time $0$ feasible, optimal forward procurement quantities</td>
</tr>
<tr>
<td>$(\cdot)^+, (\cdot)^-$</td>
<td>$\max{\cdot, 0}, -\min{\cdot, 0}$</td>
</tr>
<tr>
<td>$1{\cdot}$</td>
<td>Indicator function of event ${\cdot}$</td>
</tr>
<tr>
<td>$V$</td>
<td>Optimal forward procurement value</td>
</tr>
<tr>
<td>$V^D, V^S$</td>
<td>Demand forecast procurement policy, spot procurement policy values</td>
</tr>
<tr>
<td>$V^P$</td>
<td>Forward procurement option value</td>
</tr>
<tr>
<td>$s_d, s_f, c$</td>
<td>Time $0$ conditional standard deviations of and correlation between $\ln d$ and $\ln f$</td>
</tr>
<tr>
<td>$\phi, \Phi$</td>
<td>Standard normal density, distribution functions</td>
</tr>
<tr>
<td>$\frac{\gamma_d}{\gamma_d}$</td>
<td>Value of $s_d$ that satisfies $s_f c = \phi(z - s_d)/\Phi(z - s_d)$</td>
</tr>
<tr>
<td>$\frac{\gamma_D}{\gamma_d}$</td>
<td>Value of $s_d$ such that $V^D = V^S$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$(1 - B/A)/2$-th percentile of the standard normal distribution</td>
</tr>
<tr>
<td>$C$</td>
<td>The condition $s_f c \leq \phi(z)/\Phi(z)$</td>
</tr>
<tr>
<td>$\bar{C}$</td>
<td>Complement of $C$</td>
</tr>
<tr>
<td>$\sigma_D$</td>
<td>Log demand forecast volatility</td>
</tr>
<tr>
<td>$S(t)$</td>
<td>Time $t$ natural gas spot nominal price seasonality</td>
</tr>
<tr>
<td>$\chi(t)$</td>
<td>Natural logarithm of time $t$ deseasonalized natural gas spot nominal price</td>
</tr>
<tr>
<td>$\kappa, \xi, \sigma_\chi$</td>
<td>Speed of mean reversion, risk adjusted mean reversion level, volatility of $\chi(t)$</td>
</tr>
<tr>
<td>$dZ_D(t), dZ_\chi(t)$</td>
<td>Time $t$ standard Brownian motions increments for $D(\cdot)$ and $\chi(\cdot)$</td>
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<td>$\rho$</td>
<td>Instantaneous correlation between $dZ_D(t)$ and $dZ_\chi(t)$</td>
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<tr>
<td>$dt$</td>
<td>Infinitesimal time increment</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Overprocurement multiplier</td>
</tr>
</tbody>
</table>
normal random variable. Let \( \gamma \in \mathbb{R} \) and \( q \in \mathbb{R}_+ \). It holds that
\[
\mathbb{E}[W^{\gamma}1\{W \leq q\}] = \int_0^q w^{\gamma}g(w; \mu, \sigma)dw = e^{(\frac{\sigma^2}{2}+\mu)\gamma} \Phi \left( \frac{\ln q - (\mu + \sigma^2 \gamma)}{\sigma} \right).
\]

**Proof.** It holds that
\[
\int_0^q w^{\gamma}g(w; \mu, \sigma)dw = \int_0^q e^{\gamma \ln w}g(w; \mu, \sigma)dw = \int_{-\infty}^{\infty} e^{\gamma y} \phi(y; \mu, \sigma)dy = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{\gamma y - \frac{(y-\mu)^2}{2\sigma^2}} dy,
\]
where the second equality follows from \( g(w; \mu, \sigma) = \phi(\ln w; \mu, \sigma)/w \) and the change of variable \( y = \ln w \). By adding and subtracting \( \sigma^2 \gamma^2 + 2\sigma^2 \mu \gamma \) to the exponent of the exponential term in (21), this exponent can be written as \( [(\sigma^2 \gamma)/2 + \mu] \gamma - [(\sigma^2 \gamma + \mu)]^2/(2\sigma^2) \). Substituting this expression for this exponent yields
\[
\int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{\gamma y - \frac{(y-\mu)^2}{2\sigma^2}} dy = e^{(\frac{\sigma^2}{2}+\mu)\gamma} \int_{-\infty}^{\infty} \phi(y; \mu+\sigma^2 \gamma, \sigma)dy = e^{(\frac{\sigma^2}{2}+\mu)\gamma} \Phi \left( \frac{\ln q - (\mu + \sigma^2 \gamma)}{\sigma} \right).
\]

**Proof of Proposition 2 (Optimal forward procurement quantity with the MLN model).** The expectation in (20) can be expressed as \( \mathbb{E}[f1\{d \leq q\}] = \mathbb{E}[1\{d \leq q\}] \mathbb{E}[f|d] \). The properties of the bivariate lognormal distribution imply that under the MLN model it holds that
\[
\mathbb{E}[f|d] = Fd^{cs_f/s_d} D^{-cs_f/s_d} e^{cs_f(s_d - cs_f)/2}.
\]
This expression implies that
\[
\mathbb{E}[1\{d \leq q\}] \mathbb{E}[f|d] = FD^{-cs_f/s_d} e^{cs_f(s_d - cs_f)/2} \mathbb{E}[d^{cs_f/s_d}1\{d \leq q\}].
\]
By Lemma 1 applied to the lognormal random variable \( d|D \) with parameters \( \mu := \ln D - s_d^2/2 \), \( \sigma := s_d \), and \( \gamma := cs_f/s_d \), it follows that
\[
\mathbb{E}[d^{cs_f/s_d}1\{d \leq q\}] = D^{cs_f/s_d} e^{-cs_f(s_d - cs_f)/2} \Phi \left( \frac{\ln q - (\mu + \sigma^2 \gamma)}{\sigma} \right).
\]
Hence, (22) can be written as
\[
\mathbb{E}[1\{d \leq q\}] \mathbb{E}[f|d] = F \Phi \left( \frac{\ln q - (\mu + \sigma^2 \gamma)}{\sigma} \right).
\]
Since \( \mathbb{E}[1\{d \leq q\}] \mathbb{E}[f|d] = \mathbb{E}[fP\{d \leq q|f\}] \), substituting the right hand side of (23) into the left hand side of (4) yields
\[
\Phi \left( \frac{\ln q - (\mu + \sigma^2 \gamma)}{\sigma} \right) = \frac{1}{2} - \frac{B}{2A},
\]
which implies \( [\ln q - (\mu + \sigma^2 \gamma)]/\sigma = z \), or, equivalently, \( q = D \exp(-s_d^2/2 + cs_ds_f + zs_d) \), that is, (7) (once \( q \) is replaced with \( q^* \)). □.

Lemma 2 is useful to establish Proposition 3.
**Lemma 2** (Expected overage value with the MLN model). Consider the MLN model. Let $\mu := \ln D - s_d^2/2$, $\sigma := s_d$, and $\gamma := cs_f/s_d$. Given $q > 0$, it holds that

$$E[f(q - d)^+] = F\left[q\Phi\left(\frac{\ln q - (\mu + \sigma^2\gamma)}{\sigma}\right) - D\phi_c(s_f)\Phi\left(\frac{\ln q - (\mu + \sigma^2\gamma)}{\sigma}\right)\right].$$

**Proof.** Analogous to the proof of Proposition 2 (see the derivation of (22)), it holds that

$$E[f(q - d)^+] = E[E_{f|d}[q - d]^+] = F D F e^{cs_f(s_d - cs_f)/2} E[d^+(d^+ - d)]/2 E[d^+(q - d)^+].$$

Expanding the last expectation, applying Lemma 1 twice, and rearranging terms yields

$$E[d^+(q - d)^+] = q E[d^+(d^+ - d)] - E[d^+(d^+ - d)] = e^{\gamma / 2 + \mu} q \Phi\left(\frac{\ln q - (\mu + \sigma^2\gamma)}{\sigma}\right) - e^{\gamma / 2 + \mu} \phi_c(s_d) \Phi\left(\frac{\ln q - (\mu + \sigma^2\gamma)}{\sigma}\right) = D \phi_c(s_f)^2 / 2$$

Expression (8) follows from (26) by accounting for the transaction cost of operating in the spot market.

At optimality, the objective function of problem (1) is

$$V = (A - B) F q^* - 2 AR [f(q^* - d)^+] - (1 + A) E[f_d].$$

Applying Lemma 2 with $q = q^*$ and recalling that $\Phi(z) = 1/2 - B/(2A)$ yields

$$E[f(q^* - d)^+] = F\left[\left(\frac{1}{2} - B / 2A\right) q^* - D F C \Phi(z - s_d)\right].$$

Expression (9) follows from substituting (28) into (27), and expressing the term $E[f_d]$ by (26). Together with (2), (9) implies (10).

Inequality (11) holds because

$$\frac{V^P}{V^S} = \frac{2 A \Phi(z - s_d) K F D F}{(1 + A) K F D F} = \frac{2 A \Phi(z - s_d)}{1 + A} < \frac{A}{1 + A^*},$$

where the last step follows from $\Phi(z - s_d) \in (0, 0.5)$ and $A > 0$. □
Proof of Corollary 2 (Comparative statics of \( V_P \) with the MLN model). Parts (1)-(4) and (6) follow easily from (10). Consider part (5). The sign of the partial derivative of \( V_P \) with respect to \( s_d \) is the sign of

\[
s_f c \Phi(z - s_d) - \phi(z - s_d).
\]  

(29)

If \( c \leq 0 \), then the claimed property is true because (29) in this case is negative. If \( c > 0 \), then (29) is positive, equal to zero, and negative, respectively, whenever \( \phi(z - s_d)/\Phi(z - s_d) \) is smaller than, equal to, and greater than \( s_f c \). By Lemma 2.1 in Baricz (2008), \( \phi(\cdot)/\Phi(\cdot) \) is a strictly decreasing function. This implies that \( \phi(z - s_d)/\Phi(z - s_d) \) is a strictly increasing function of \( s_d \).

Moreover, it holds that \( \lim_{s_d \to 0} \phi(z - s_d)/\Phi(z - s_d) = \phi(z)/\Phi(z) \). Therefore, if \( s_f c > \phi(z)/\Phi(z) \), that is, \( \bar{C} \) is true, then the threshold \( \bar{s}_d \) exists and (29) is positive, equal to zero, and negative, respectively, when \( s_d \in (0, \bar{s}_d) \), \( s_d = \bar{s}_d \), and \( s_d \in (\bar{s}_d, \infty) \); if \( s_f c \leq \phi(z)/\Phi(z) \), that is, \( C \) is true, then (29) is negative for all \( s_d > 0 \). □

Proof of Proposition 4 (Performance of the demand forecast policy with the MLN model). (1) Suppose that \( q^* > D \). This occurs when and only when \( s_d \in (0, 2\bar{s}_d) \). Define \( \mu := \ln D - s_d^2/2 \), \( \sigma := s_d \), and \( \gamma := cs_f/s_d \). It follows from Lemma 2 and the equality \( \Phi(z) = (1 - B/A)/2 \) that

\[
V - V^D = (A - B)(q^* - D)F + 2A \left[ \Phi \left( \frac{\ln D - (\mu + \sigma^2\gamma)}{\sigma} \right) D - K^C D \Phi \left( \frac{\ln D - (\mu + \sigma^2\gamma) - \sigma}{\sigma} \right) \right] F
\]

\[
-2A \left[ \Phi \left( \frac{\ln q^* - (\mu + \sigma^2\gamma)}{\sigma} \right) q^* - K^C D \Phi \left( \frac{\ln q^* - (\mu + \sigma^2\gamma) - \sigma}{\sigma} \right) \right] F
\]

\[
< (A - B)(q^* - D)F + 2A \Phi \left( \frac{\ln q^* - (\mu + \sigma^2\gamma)}{\sigma} \right) (D - q^*) + 2A \left[ \Phi \left( \frac{\ln q^* - (\mu + \sigma^2\gamma) - \sigma}{\sigma} \right) - \Phi \left( \frac{\ln D - (\mu + \sigma^2\gamma) - \sigma}{\sigma} \right) \right] K^C FD
\]

\[
= (A - B)(q^* - D)F - 2A \Phi(z)F(q^* - D)
\]

\[
+2A \left[ \Phi(z - s_d) - \Phi \left( \frac{-s_d}{2} - cs_f \right) \right] K^C FD
\]

\[
= (A - B)(q^* - D)F - (A - B)F(q^* - D)
\]

\[
+2A \left[ \Phi(z - s_d) - \Phi \left( \frac{-s_d}{2} - cs_f \right) \right] K^C FD
\]

\[
= 2A \Phi(z - s_d)K^C FD - 2A \Phi \left( \frac{-s_d}{2} - cs_f \right) K^C FD,
\]

(30)

which is the difference between the suboptimality of the spot procurement policy and a positive number.
Suppose that \( s_d \in [2\pi_d^q, \infty) \), so that \( q^* \leq D \). The claimed property holds when \( s_d = 2\pi_d^q \) because in this case the demand forecast policy is optimal; in particular, this policy strictly dominates the spot procurement policy for this value of \( s_d \). Assume that \( s_d > 2\pi_d^q \). The demand forecast policy outperforms the spot procurement policy provided that \( V^S - V^D < 0 \). Expression (12) implies that

\[
V^S - V^D = -\left\{ A - B - 2A \left[ \Phi \left( \frac{sd} {2} - cs_f \right) - e^{cs_ds_f} \Phi \left( -\left( \frac{sd} {2} + cs_f \right) \right) \right] \right\} FD. \tag{31}
\]

Because when \( s_d = 2\pi_d^q \) the demand forecast policy is optimal and strictly dominates the spot procurement policy, (31) is negative for this value of \( s_d \). We now show that (31) increases when \( s_d \) increases past \( 2\pi_d^q \).

The function \( \Phi(s_d/2 - cs_f) \) increases in \( s_d \). The derivative of \( -\exp(cs_ds_f)\Phi(-s_d/2 - cs_f) \) with respect to \( s_d \) is

\[
e^{cs_ds_f} \left[ \frac{1}{2} \phi \left( \frac{s_d} {2} - cs_f \right) - \Phi \left( -\frac{s_d} {2} - cs_f \right) \right],
\]

and is positive provided that

\[
\frac{\phi(-s_d/2 - cs_f)}{\Phi(-s_d/2 - cs_f)} > 2cs_f. \tag{32}
\]

The left hand side of (32) strictly increases in \( s_d \), because \( \phi(\cdot)/\Phi(\cdot) \) is a strictly decreasing function (Baricz 2008, Lemma 2.1). Hence, the function \( -\exp(cs_ds_f)\Phi(-s_d/2 - cs_f) \) may decrease when \( s_d \) increases up to some value but eventually strictly increases in \( s_d \). Therefore, there exists a value \( \overline{s}_d^D \in (2\pi_d^q, \infty) \), which depends on the problem parameters, such that the demand forecast policy outperforms the spot procurement policy when \( s_d \in (2\pi_d^q, \overline{s}_d^D) \).

(2) Suppose that \( q^* < D \), which is equivalent to \( s_d \in (2\pi_d^q, \infty) \). Also suppose that \( s_d \leq 2cs_f \), so that \( \Phi(s_d/2 - cs_f) \leq 1/2 \). Following the approach used to establish part (1) of this proposition yields

\[
V - V^D = (A - B)(q^* - D)F + 2A \left[ \Phi \left( \frac{\ln D - (\mu + \sigma^2 \gamma)}{\sigma} \right) D - K^C D \Phi \left( \frac{\ln D - (\mu + \sigma^2 \gamma)}{\sigma} \right) \right] F
- 2A \left[ \Phi \left( \frac{\ln q^* - (\mu + \sigma^2 \gamma)}{\sigma} \right) q^* - K^C D \Phi \left( \frac{\ln q^* - (\mu + \sigma^2 \gamma)}{\sigma} \right) \right] F
< (A - B)(q^* - D)F + 2A \left[ \Phi \left( \frac{\ln D - (\mu + \sigma^2 \gamma)}{\sigma} \right) D - K^C D \Phi \left( \frac{\ln q^* - (\mu + \sigma^2 \gamma)}{\sigma} \right) \right] F
- 2A \left[ \Phi \left( \frac{\ln q^* - (\mu + \sigma^2 \gamma)}{\sigma} \right) q^* + K^C D \Phi \left( \frac{\ln q^* - (\mu + \sigma^2 \gamma)}{\sigma} \right) \right] F
= (A - B)(q^* - D)F + 2A \Phi \left( \frac{sd}{2} - cs_f \right) DF - 2A \Phi(z)q^* F
\]

OA-6
\[ (A - B)(q^* - D)F + ADF - (A - B)q^*F \leq BDF. \]  

(33)

Since \( \Phi(z - s_d) \in (0, 0.5) \), it holds that

\[ |V| = [1 + A - 2A\Phi(z - s_d)] K^C FD > (1 + A - 2A \cdot 0.5) K^C FD = K^C FD. \]  

(34)

Thus, (33) and (34) imply that if \( s_d \in (2(z + cs_f)^+, 2cs_f] \) then

\[ \frac{V - V^D}{|V|} < \frac{B}{K^C}. \]  

(35)

C. Numerical Study with a Variant of the Base Model

This section presents some numerical results with a variant of our base model in which the forward nominal price, \( F \), differs from the expected spot nominal price, \( \mathbb{E}[f] \), that is, condition (3) does not hold. This analysis provides some support for the robustness of our insights, even in some situations when the forward nominal price exceeds the expected spot nominal price.

The avoid arbitrage opportunities when condition (3) is not satisfied, we assume that both the strategy of buying forward and selling spot and the strategy of selling forward and buying spot one unit of commodity have negative values, that is:

\[ (1 + B)F > (1 - A)\mathbb{E}[f], \]

\[ (1 - B)F < (1 + A)\mathbb{E}[f]. \]

These inequalities imply the following no-arbitrage bounds for the ratio \( \mathbb{E}[f]/F \):

\[ \frac{1 - B}{1 + A} < \frac{\mathbb{E}[f]}{F} < \frac{1 + B}{1 - A}. \]  

(36)

When (36) holds, it is easy to generalize the optimality condition (4) for our base model, as well as various formulas of interest under the MLN model. For brevity, we omit these generalizations.

The ensuing discussion summarizes the application of these generalized formulas to the instances presented in §5.1 using a slight modification of the initial condition for the price model (16) so that the ratio \( \mathbb{E}[f]/F \) can also take on values different from 1. Performing this analysis requires specifying a value for this ratio. Given the bounds (36), there are two cases to consider: (1) \( \mathbb{E}[f]/F \in [1, (1 + B)/(1 - A)) \) and (2) \( \mathbb{E}[f]/F \in ((1 - B)/(1 + A), 1] \) (1 is present in both intervals so that the base case \( \mathbb{E}[f]/F = 1 \) is included in both cases).
Case (1). We vary the values of the ratio $E[f]/F$ in discretized subsets of the intervals $[1,(1 + B)/(1 - A))$ corresponding to each of the values of the forward transaction cost, $B$, considered in §5.1. These subsets are $\{1.00,1.01,1.02,1.03\}$, $\{1.00,1.01,1.02,1.03,1.04\}$, and $\{1.00,1.01,1.02,1.03,1.04,1.05,1.06\}$ when the value of $B$ is 0.025%, 0.25%, and 2.5%, respectively. Table 7 reports these results. The value of the forward procurement option obviously increases relative to the case when the ratio $E[f]/F$ is equal to 1. This increase can be substantial. The demand forecast forward procurement policy continues to be near optimal.

Case (2). It is easy to verify that the value of the forward procurement option tends to zero when the ratio $E[f]/F$ approaches $(1 - B)/(1 + A)$, obviously from above. Consequently, the forward procurement option can retain substantial value only if the ratio $E[f]/F$ is above a threshold in the interval $((1 - B)/(1 + A),1]$. This threshold depends on the values of the parameters that define each instance. Hence, there could be a separate threshold for each such instance, which would make it impossible to separately report here all these thresholds and the corresponding results. We thus use the following conservative reporting approach: Report the results obtained by varying the ratio $E[f]/F$ in an interval such that the value of the forward...

Table 7: Summary of the numerical results for case (1).

<table>
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<th>Range</th>
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<tbody>
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<td>$1,850,447-4,408,565$</td>
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<tr>
<td>$V^P/</td>
<td>V^S</td>
</tr>
<tr>
<td>$q^*/D$</td>
<td>95.58-140.39%</td>
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<td>$V - V^D$</td>
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<td>$(V - V^D)/</td>
<td>V</td>
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<table>
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<th>Range</th>
</tr>
</thead>
<tbody>
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<td>$1,712,886-5,000,504$</td>
</tr>
<tr>
<td>$V^P/</td>
<td>V^S</td>
</tr>
<tr>
<td>$q^*/D$</td>
<td>93.49-181.19%</td>
</tr>
<tr>
<td>$V - V^D$</td>
<td>$1-577,335$</td>
</tr>
<tr>
<td>$(V - V^D)/</td>
<td>V</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^P$</td>
<td>$502,975-4,830,948$</td>
</tr>
<tr>
<td>$V^P/</td>
<td>V^S</td>
</tr>
<tr>
<td>$q^*/D$</td>
<td>72.13-155.86%</td>
</tr>
<tr>
<td>$V - V^D$</td>
<td>$88-450,344$</td>
</tr>
<tr>
<td>$(V - V^D)/</td>
<td>V</td>
</tr>
</tbody>
</table>
Table 8: Summary of the numerical results for case (2).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Range</th>
<th>( B = 0.025% ) and ( \mathbb{E}[f]/F \in {0.9725, 0.9750, \ldots, 1.0000} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V^P )</td>
<td>$333,452-2,366,699</td>
<td></td>
</tr>
<tr>
<td>( V^P/</td>
<td>V^S</td>
<td>)</td>
</tr>
<tr>
<td>( q^*/D )</td>
<td>67.82-100.43%</td>
<td></td>
</tr>
<tr>
<td>( V - V^D )</td>
<td>$0-319,255</td>
<td></td>
</tr>
<tr>
<td>( (V - V^D)/</td>
<td>V</td>
<td>)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Range</th>
<th>( B = 0.25% ) and ( \mathbb{E}[f]/F \in {0.9750, 0.9800, \ldots, 1.0000} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V^P )</td>
<td>$348,449-2,221,029</td>
<td></td>
</tr>
<tr>
<td>( V^P/</td>
<td>V^S</td>
<td>)</td>
</tr>
<tr>
<td>( q^*/D )</td>
<td>68.25-100.01%</td>
<td></td>
</tr>
<tr>
<td>( V - V^D )</td>
<td>$1-313,434</td>
<td></td>
</tr>
<tr>
<td>( (V - V^D)/</td>
<td>V</td>
<td>)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Range</th>
<th>( B = 2.5% ) and ( \mathbb{E}[f]/F \in {0.9965, 0.9970, \ldots, 1.0000} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V^P )</td>
<td>$338,825-780,464</td>
<td></td>
</tr>
<tr>
<td>( V^P/</td>
<td>V^S</td>
<td>)</td>
</tr>
<tr>
<td>( q^*/D )</td>
<td>67.73-97.83%</td>
<td></td>
</tr>
<tr>
<td>( V - V^D )</td>
<td>$14,562-328,492</td>
<td></td>
</tr>
<tr>
<td>( (V - V^D)/</td>
<td>V</td>
<td>)</td>
</tr>
</tbody>
</table>

The value of the forward procurement option is at least 0.5% of the absolute value of the spot procurement policy on all the instances corresponding to each given value of the forward transaction cost, \( B \); that is, we take 0.5% as the minimum “interesting” value of this option. In other words, we focus on reporting the maximum threshold across all the instances corresponding to a given value of the forward transaction cost, \( B \). Hence, the true threshold can be lower for some of our instances, which is why our reporting approach is conservative.

The maximum threshold is about 0.9725, 0.975, and 0.9965 when the forward transaction cost, \( B \), is equal to 0.025\%, 0.25\%, and 2.5\%, respectively. The corresponding results are obtained by varying the ratio \( \mathbb{E}[f]/F \) between 0.9725 and 1.0000, 0.975 and 1.000, and 0.9965 and 1.0000 in increments of 0.0025, 0.005, and 0.0005, respectively. Table 8 reports these results. The value of the forward procurement option obviously decreases relative to the case when the ratio \( \mathbb{E}[f]/F \) is equal to 1. However, this option can remain valuable. The suboptimality of the demand forecast forward procurement policy can be small.
D. Extended Model and Its Analysis

This section extends our base model in its aggregation of dates interpretation by disaggregating these dates (§D.1); partially characterizes the optimal policy of this extended model and provides a formula to compute its value under an extension of the MLN model (§D.2); and conducts a numerical study of this policy and its value similar to the one performed with our base model (§D.3).

D.1 Formulation

In our extended model, the aggregate time period $T$ is subdivided into $I$ equally spaced dates, each denoted by time $T_i$ with $i \in \mathcal{I} := \{1, \ldots, I\}$. For example, if the time period $T$ represents a month, then $I$ is the number of days in this month and the date $T_i$ is the $i$-th day in this month. There is still a single purchase at time 0 in the forward market. Purchasing an amount of supply $q$ in the forward market at this time entails a payment equal to $(1 + B)Fq$ at time $T_1$ (recall that $F$ is the forward nominal price at time 0). Different from the base model, a fraction $1/I$ of the quantity $q$ purchased at time 0 is delivered on each date $T_i$. That is, our base model corresponds to the special case in which $I = 1$.

We continue to use an equivalent measure under which the forward nominal price process $F(\cdot)$ is a martingale during the time interval $(0, T_1]$. Since there are now multiple cash flows on different dates during the delivery period, we explicitly model the risk free discount factor. We denote by $\delta$ the risk free discount factor from time $T_i$ back to time $T_{i-1}$ for each $i \in \mathcal{I} \setminus \{1\}$. We let $f_i$ be the spot nominal price at time $T_i$. We extend the spot and forward nominal price convergence condition stated in §3, that is, $F(T) \equiv f$. Specifically, at the beginning of the delivery period, time $T_1$, the forward nominal price converges to the fraction $1/I$ of the sum of the discounted expected values of the spot nominal prices during the delivery period:

$$F(T_1) \equiv \frac{1}{I} \sum_{i \in \mathcal{I}} \delta^{i-1} \mathbb{E}_1[f_i],$$

where $\mathbb{E}_1$ is expectation conditional on the information available at time $T_1$.

Demand is date specific. The spot demand on date $T_i$ is the quantity $d_i$. At time 0, the firm’s forecast of the total demand on dates $T_1$ through $T_I$ is $D$. The firm’s time 0 forecast of the demand on date $T_i$ is the fraction $\beta_i$ of the total demand forecast; that is,

$$\mathbb{E}[d_i \mid D] = \beta_i D,$$
with $\beta_i \geq 0$ for all $i \in I$ and $\sum_{i \in I} \beta_i = 1$. The firm must fully satisfy demand on each date $T_i$. Due to the lack of storage, the firm can do this either by employing the amount of the quantity procured forward and delivered on each date, $q/I$, or by procuring in the spot market on the date. Excess supply delivered on a given date is sold into the spot market. That is, our extended model includes multiple spot make up transactions. For simplicity, we assume that the spot transaction costs are constant across the dates $T_1$ through $T_I$. That is, the purchase and sale spot prices on date $T_i$ are $(1 + A)f_i$ and $(1 - A)f_i$, respectively. Each spot demand and nominal price are correlated random variables. Analogous to our base model, we assume that each spot demand random variable is a continuous and nonnegative random variable.

The firm must decide at time 0 the quantity to purchase forward for delivery at times $T_1$ through $T_I$. That is, at time 0 the firm needs to solve the following optimization problem:

$$V := \max_{q \geq 0} \sum_{i \in I} \delta_i^{-1} \mathbb{E} \left[ (1 - A)f_i \left( \frac{q}{I} - d_i \right)^+ - (1 + A)f_i \left( \frac{q}{I} - d_i \right)^- \right] - (1 + B)Fq.$$  \hspace{1cm} (39)

This optimization problem assumes that all cash flows are valued as of time $T_1$. Problem (39) reduces to problem (1) when $I = 1$.

The spot procurement policy in this extended setting corresponds to procuring the realized spot demand on each date in the delivery period. The value of this policy is thus $V^S = -\sum_{i \in I} \delta_i^{-1} \mathbb{E} [(1 + A)f_id_i]$. The value of the forward procurement option $V^P$ remains the difference $V - V^S$.

**D.2 Structural Analysis**

We focus on characterizing the structure of the optimal forward procurement policy and its value. We start this analysis by pointing out that condition (37) implies that the forward nominal price at time 0, $F$, satisfies

$$F = \frac{1}{I} \sum_{i \in I} \delta_i^{-1} \mathbb{E}[f_i].$$  \hspace{1cm} (40)

This is true because

$$F = \mathbb{E}[F(T)] = \mathbb{E} \left[ \frac{1}{I} \sum_{i \in I} \delta_i^{-1} \mathbb{E}_1[f_i] \right] = \frac{1}{I} \sum_{i \in I} \delta_i^{-1} \mathbb{E} [\mathbb{E}_1[f_i]] = \frac{1}{I} \sum_{i \in I} \delta_i^{-1} \mathbb{E}[f_i],$$

where the first equality is due to our use of an equivalent measure, under which the forward nominal price is a martingale, the second equality follows from condition (37), and the last equality holds by the law of iterated expectations.
We use condition (40) in Proposition 5 to characterize the optimality condition of problem (39). To make this characterization more specific, we also extend our MLN model to the setting of this section. We assume that the log spot demand and nominal price on each date $T_i$ are jointly normally distributed with respective standard deviations $s_{d,i}$ and $s_{f,i}$, respective means $\ln(\beta_i D) - s_{d,i}^2/2$ and $m_i$, and correlation coefficient $c_i$. This model satisfies condition (38). We also assume that the parameters $m_i$’s, $s_{f,i}$’s, and $F$ satisfy condition (40).

Proposition 5 provides partial characterizations of the optimal forward procurement quantity and expressions for the values of the spot procurement policy, the optimal forward procurement policy, and the forward procurement option under our extended MLN model. We extend $K^C$ and $\alpha$ as $K^C_i := \exp(c_is_{d,i}s_{f,i})$ and $\alpha_i(q) := 2A\Phi(\ln((q/I)/(\beta_i D))/s_{d,i} - c_is_{f,i} - s_{d,i}/2)$.

**Proposition 5 (Extended model).** (1) A positive solution to problem (39) is optimal if and only if it satisfies the condition

$$\sum_{i \in I} \delta^{-1}\mathbb{E}\left[\frac{f_i}{F}\mathbb{P}\{d_i \leq \frac{q}{I} | f_i\}\right] = \frac{1}{2} \left(1 - \frac{B}{A}\right).$$

If no positive solution satisfies this equation, then zero optimally solve problem (39).

(2) Under the extended MLN model, the optimal solution to problem (39) is positive and satisfies the condition

$$\sum_{i \in I} \left\{\frac{\delta^{-1}\mathbb{E}[f_i]}{F}\right\} \Phi\left(\frac{\ln((q/I)/(\beta_i D)) - c_is_{f,i} + s_{d,i}/2}{s_{d,i}}\right) = \frac{1}{2} \left(1 - \frac{B}{A}\right).$$

(3) Moreover, the values of the spot procurement policy, the optimal procurement policy, and the forward procurement option are

$$V^S = -(1 + A)D \sum_{i \in I} \delta^{-1}\mathbb{E}[f_i],$$

$$V = V^S + \sum_{i \in I} \alpha_i(q^*)K^C_i \beta_i D\delta^{-1}\mathbb{E}[f_i],$$

$$V^P = \sum_{i \in I} \alpha_i(q^*)K^C_i \beta_i D\delta^{-1}\mathbb{E}[f_i].$$

**Proof.** (1) Exploiting condition (40), the objective function of problem (39) can be rewritten as

$$(B - A)Fq - 2A \sum_{i \in I} \delta^{-1}\mathbb{E}\left[f_i\left(\frac{q}{I} - d_i\right)^+\right] - (1 + A) \sum_{i \in I} \delta^{-1}\mathbb{E}[f_i d_i].$$

It is easy to verify that (46) is proper concave in $q$ and tends to $-\infty$ as $q$ tends to $\infty$. The first order optimality condition (41), which holds by (40), is thus sufficient and necessary for the
optimality of a positive solution to problem (39). If no positive solution satisfies this condition, then zero must optimally solve this problem.

(2) Condition (42) follows from applying Lemma 1 in a manner similar to its application in the proof of Proposition 2. This condition is always satisfied at a positive value of \( q \), as the left hand side of (42) tends to 0 and \( I > (1 - B/A)/2 \), respectively, when \( q \) tends to 0 and \( \infty \) (in the latter case this is due to (40)).

(3) Expression (43) follows from letting \( q \) be equal to 0 in (46) and using the properties of the lognormal distribution to express each term \( E[f_id_i] \). Expression (44) follows from applying Lemma 2 to express each term \( E[f_i(q/I - d_i)^+] \) with \( q = q^* \) in (46), exploiting the optimality condition (42), and using (43). Expression (45) follows immediately from (43) and (44). \( \square \)

Proposition 5 generalizes Proposition 1 and partially Propositions 2 and 3. Extending the first order optimality condition (4), condition (41) includes in its left hand side the sum of the discounted weighted overage probabilities, one for each date in the delivery period, and in its right hand side the critical ratio \( (1 - B/A)/2 \). Under our extended MLN model, this first order optimality condition reduces to the condition (42). In general, this condition does not admit a closed form expression, as is instead possible with the base model under the MLN model, because each term in the sum in its left hand side involves the standard normal cumulative distribution evaluated at different values. The expressions (43), (44), and (45) for the values of the spot procurement policy, the optimal forward procurement policy, and the forward procurement option are qualitatively consistent with their respective analogous expressions (8), (9), and (10).

As in our analysis of our base model, we consider the demand forecast forward procurement policy. By (46), Lemma 2 in Online Appendix B, and (40), the value of this policy with the extended MLN model is

\[
V^D = V^S + (A - B)FD - 2AD \sum_{i \in I} \left[\frac{1}{T} \Phi \left( -\ln(I \beta_i) + s_{d,i}/2 - c_i s_{f,i} \right) - \beta_i K^C_i \Phi \left( -\ln(I \beta_i) - (s_{d,i}/2 + c_i s_{f,i}) \right) \right] \delta^{i-1} E[f_i] .
\]

Moreover, if \( \beta_i = 1/I \) for all \( i \in I \), then (47) simplifies to

\[
V^D = V^S + (A - B)FD - \frac{2AD}{T} \sum_{i \in I} \left[ \Phi \left( s_{d,i}/2 - c_i s_{f,i} \right) - K^C_i \Phi \left( -(s_{d,i}/2 + c_i s_{f,i}) \right) \right] \delta^{i-1} E[f_i] .
\]

We use (48) in §D.3.
D.3 Numerical Study

We extend our numerical study discussed in §5 by taking the time interval \([T_1, T_I]\) to be the month of February 2011. We interpret the parameter \(I\) as the number of days in this month, and thus set it equal to 28. We let the risk free discount rate to be 0.01 and define \(\delta := \exp(-0.01/365)\).

We specify the extended MLN model in a manner similar to our specification of the MLN model in §5.1. We continue to model the evolution of the log deseasonalized spot nominal price \(\chi(t)\) as a mean reverting model. As in Lai et al. (2011, Table 2), we set the parameter \(\xi\) equal to \(-2.0421\) and each parameter \(S(T_i)\) equal to 1.0761 (the February seasonality parameter, a constant). We then use condition (40) to determine the spot nominal price \(f(0)\). That is, under model (16) the mean and standard deviations of the time \(T_1\) log deseasonalized spot nominal price \(\chi(T_i)\) satisfy, respectively,

\[
\begin{align*}
m_i &= \chi_0 e^{-\kappa T_i} + \xi (1 - e^{-\kappa T_i}), \\
s_{f,i} &= \sigma \sqrt{1 - e^{-2\kappa T_i}}.
\end{align*}
\]

It follows that

\[
\mathbb{E}[f_i | f(0)] = S(T_i) \exp \left( \chi(0)e^{-\kappa T_i} + \xi (1 - e^{-\kappa T_i}) + \frac{\sigma^2}{4\kappa} (1 - e^{-2\kappa T_i}) \right).
\tag{49}
\]

For a given set of parameters, we thus solve the equation

\[
F = \frac{1}{I} \sum_{i \in \mathcal{I}} \delta_i \exp \left( \chi(0)e^{-\kappa T_i} + \xi (1 - e^{-\kappa T_i}) + \frac{\sigma^2}{4\kappa} (1 - e^{-2\kappa T_i}) \right)
\tag{50}
\]

in the unknown \(\chi(0)\). Equation (50) always has a unique solution, as its right hand side tends to 0 when \(\chi(0)\) tends to \(-\infty\) and it monotonically increases in \(\chi(0)\).

We obtain the distribution of the spot demand for time \(T_i\) by modeling the evolution of the forecast of this demand, which we denote by \(D_i(\cdot)\), in a manner analogous to how we model the evolution of the aggregate demand forecast \(D(\cdot)\) in §5.1. Specifically, we let each \(D_i(\cdot)\) evolve during the time interval \([0, T_i]\) as a driftless geometric Brownian motion correlated with the log deseasonalized spot nominal price process. That is, for all \(i \in \mathcal{I}\) we assume that

\[
\begin{align*}
dD_i(t) &= \sigma_i D_i(t) dZ_i(t), \\
dZ_i(t) dZ_{i'}(t) &= \rho_i dt,
\end{align*}
\tag{51}
\tag{52}
\]

where \(\sigma_i\) is the volatility of the log spot demand forecast for date \(T_i\), \(dZ_i(t)\) is an increment to a standard Brownian motion specific to date \(T_i\), and \(\rho_i\) is the instantaneous correlation coefficient.

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between this standard Brownian motion and the one associated with the log deseasonalized spot nominal price in (16).

Under model (16) and (51)-(52), the standard deviation of the log spot demand is

\[ s_{d,i} = \sigma_i \sqrt{T_i}, \]

and the correlation between this random variable and the log deseasonalized spot nominal price is

\[ c_i = \frac{\rho_i (1 - e^{-\kappa T_i})/\kappa}{\sqrt{T_i(1 - e^{-2\kappa T_i})/(2\kappa)}}. \]

For simplicity, we set the numerical values of each of the parameters \( \sigma_i \) and \( \rho_i \) equal to the ones of the parameters \( \sigma_D \) and \( \rho \) considered in §5.1. Also for simplicity, we assume that each parameter \( \beta_i \) is equal to \( 1/I = 1/28 \). We have considered different choices of these parameters obtaining consistent results.

Our numerical results with the extended model, summarized in Table 9, mirror the ones obtained with the base model. Specifically, the value of the forward procurement option is $2,282,194 per month in the base case and ranges from $409,983 to $2,353,984 per month across all the considered instances. These figures correspond to improvements of 3.39% and 0.61-3.49%, respectively, on the cost of the spot procurement policy. In the base case, the optimal forward procurement quantity is 99.94% of the demand forecast \( D \), that is, the demand forecast for the entire time period \([T_1, T_I]\). Across all our instances, the optimal forward procurement quantity varies in between 62.72% and 100.80% of this forecast. This range is 92.19-100.80%, 89.50-100.12%, and 62.72-97.18% when the forward transaction cost is 0.025%, 0.25%, and 2.5%, respectively. The demand forecast procurement policy is near optimal, as its suboptimality is no more than 0.56%, that is, $372,086 per month. These results suggest that the insights into the value of the forward procurement option, the relationship between the optimal and the demand forecast procurement policies, and the performance of the latter policy obtained with our base model have broader applicability, as they carry through to the setting of our extended model.
References

