Real Options and Merchant Operations of Energy and Other Commodities

Nicola Secomandi\textsuperscript{1} and Duane J. Seppi\textsuperscript{2}

\textsuperscript{1} Tepper School of Business, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213-3890, USA, ns7@andrew.cmu.edu
\textsuperscript{2} Tepper School of Business, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213-3890, USA, ds64@andrew.cmu.edu

Abstract

The value chain for energy and other commodities entails physical conversions through refineries, power plants, storage facilities, and transportation and other capital-intensive infrastructure. When the operation of such commodity conversion assets occurs alongside liquid markets for the input and output commodities, the operating flexibility of conversion assets can be managed as real options on the underlying commodity prices. Merchant operations is an integrated trading and operations approach that (i) buys and sells commodities to support market-value maximizing optimal, or at least near-optimal, operating policies and (ii) values conversion assets, for capital budgeting and trading purposes, based on the cash flows such policies produce. This monograph provides a unique integrated finance and operations perspective on the topic of merchant operations. In particular, this monograph introduces the concept of merchant operations; presents the basic prin-
ciples of option valuation; surveys foundational models of commodity and energy price evolution; analyzes the structure of optimal operating policies for commodity conversions, focusing specifically on inventory and other intertemporal linkages in storage, inventory acquisition and disposal, and swing assets; considers a variety of heuristic storage operating policies; and discusses future trends in this multidisciplinary area of research and business applications.
# Contents

1. **Aim and Scope**

2. **Commodity Conversion Assets and Merchant Operations**
   
   2.1 Commodities and Energy Sources
   
   2.2 Physical and Financial Markets
   
   2.3 Conversion Assets and Merchant Operations
   
   2.4 Commodity Conversion Assets as Real Options
   
   2.5 Outline of this Monograph
   
   2.6 Notes

3. **Commodity Option Valuation**
   
   3.1 Introduction
   
   3.2 Statically Complete Markets
   
   3.3 Dynamically Complete Markets
   
   3.4 Dynamically Incomplete Markets
   
   3.5 Calibration
Contents

3.6 Summary 40
3.7 Notes 41

4 Modeling Commodity and Energy Prices 43
4.1 Introduction 43
4.2 Spot-price Evolution Models 52
4.3 Futures Term Structure Models 61
4.4 Equilibrium Models 67
4.5 Empirical Research on Commodity Prices 69
4.6 Summary 71
4.7 Notes 71

5 Modeling Commodity Storage Assets 73
5.1 Introduction 73
5.2 Model 78
5.3 Basestock Optimality 83
5.4 Price Monotonicity of the Optimal Basestock Targets 92
5.5 Computation 95
5.6 The Value of Optimally Interfacing Operational and Trading Decisions 97
5.7 Conclusions 104
5.8 Notes 104

6 Benchmarking the Practice Based Management of Commodity Storage Assets 107
6.1 Introduction 108
6.2 Valuation Problem and Exact SDP 111
6.3 Practice-based Heuristics 114
6.4 ADP Model 118
6.5 Upper Bounds 123
6.6 Numerical Results 124
6.7 Conclusions 131
6.8 Notes 132
7 Modeling Other Commodity Conversion Assets 135

7.1 Introduction 135
7.2 An Inventory Disposal Asset 136
7.3 An Inventory Acquisition Asset 139
7.4 Swing Assets 141
7.5 Conclusions 144
7.6 Notes 144

8 Trends 147

8.1 Financial Hedging 147
8.2 Analysis of Other Commodity Conversion Assets 148
8.3 Approximate Dynamic Programming 149
8.4 Price Model Error 150
8.5 Price Impact 150
8.6 Equilibrium Asset Pricing 151
8.7 Capacity Choice 151

Acknowledgements 153

References 154
Commodity conversion assets perform various transformation processes, including the production, refining, industrial and commercial consumption, and distribution of physical commodities and energy sources, such as grains, metals, electricity, coal, crude oil, and natural gas. This monograph deals with the management and valuation of conversion assets. Commodities and energy are traded on physical markets. It is thus natural to approach the management of commodity conversion assets from a merchant perspective, which adjusts the level of the conversion activities to profit from the dynamics of commodity prices. We introduce the expression *merchant operations* to describe this approach.

Implementing merchant operations to maximize the market value of commodity conversion assets is a complex task. It requires both models of the evolution of commodity prices and stochastic optimization models of the conversion activities. The existence of traded contracts on commodities and energy sources allows commodity conversion assets to be interpreted as real options on the prices of the underlying commodities. This real option interpretation greatly facilitates formulating the necessary mathematical models.
The valuation of real options shares the same theoretical foundations as the valuation of financial options. However, the real options that arise in the context of merchant operations are distinguished from financial options by one or more of the following features: (i) decisions at multiple dates, (ii) intertemporal linkages across decisions, (iii) payoffs determined by operational costs and contractual provisions, (iv) engineering-based constraints on operating decisions, and (v) quantity decisions rather than binary exercise/no-exercise decisions. The valuation of American and Bermudan financial options also involves decisions at multiple dates and intertemporal linkages across decisions, but the other features in this list are largely unique to merchant operations. Moreover, even when the structure of an optimal operating (exercise) policy can be explicitly characterized, determining such a policy typically involves numerical computation and approximations. Closed-form solutions are rare in merchant operations, while they are more widespread for financial options.

The aim of this monograph is to present the basic tenets of merchant operations, that is, the management of commodity conversion assets as real options on commodity prices. The scope of this monograph is on the foundational principles that underlie the valuation of real options, on basic models of both the evolution of commodity and energy prices and commodity conversion assets, and on optimal and heuristic operating policies with a focus on commodity storage assets.

A unique aspect of merchant operations is the integration of financial and operational management aspects. This monograph reflects this integrated perspective. Chapter 2 introduces commodity conversion assets and merchant operations. Chapters 3 and 4 discuss the valuation of real options and models of the evolution of commodity and energy prices, respectively. Chapters 5 and 6 delve into the operational management of commodity storage assets. Chapter 7 deals with inventory disposal/acquisition and swing assets. Future trends in merchant operations research and applications are discussed in Chapter 8.
This chapter briefly introduces the basic commodity groups in §2.1, and illustrates the trading of commodities in physical and financial markets in §2.2. This discussion provides the necessary elements for introducing the concepts of commodity conversion assets and merchant operations in §2.3. This chapter also shows that merchant operations can be usefully cast in the framework of real options in §2.4, providing some examples of the real option management of commodity conversion assets. Some of these examples are substantial simplifications of how commodity conversion assets are operated in practice, but they set the stage for more realistic models. The discussion of these examples provides a point of departure for the rest of this monograph, which is outlined in §2.5. Section 2.6 offers pointers to the existing literature.

2.1 Commodities and Energy Sources

According to the Webster’s New Universal Unabridged Dictionary [218], a commodity is “any unprocessed or partially processed good.” Commodities can be grouped according to three basic categories: agricultural, metals, and energy sources. Each of the groups includes sev-
eral commodity types, for example:

- **Agriculturals**: grains (corn, oats, rice, and wheat), oil and meal (soybean, soyoil, and soymeal), livestock (pork and beef), foodstuff (cocoa, coffee, orange juice, potatoes, and sugar), textiles (cotton), and forest products (lumber and pulp).
- **Metals**: gold, silver, platinum, palladium, copper, and aluminum.
- **Energy sources**: coal, crude oil, heating oil, gasoline, natural gas, propane gas, and electricity.

Commodity types can be further categorized according to grade and quality. In addition, due to limited transportation and storage capacity, the same commodity type at different locations and/or times effectively constitutes separate commodities. For instance, consider natural gas at two ends of a pipeline that transports it, or the availability of natural gas in a storage facility at two different dates. Thus, location and time are defining attributes of commodity types.

Commodities are basic inputs to production, distribution, and consumption activities. They thus play important economic roles. For instance, natural gas is extensively used for heating (e.g., by about 50% of the households in the United States; Casselman [49]) and electricity generation (e.g., about 30% of worldwide electricity production is based on natural gas; Geman [99]). It is increasingly the fuel of choice for new electricity generation projects, due in part to the recent shale gas boom (Smith [198]). It is having an impact on the worldwide liquefied natural gas trade (Davis and Gold [65]) and the planning of new major pipeline construction (Chazan [55], Gold [106]). Natural gas is also an input to several manufacturing processes, including the refining of oil, and the production of chemicals, ammonia and methanol, steel and aluminum, and paper (Kaminski and Prevatt [128]).

Although energy is just one commodity group, it is emphasized in this monograph because of its economic importance. There is also a substantial literature that deals specifically with the application of real option methods to energy.
2.2 Physical and Financial Markets

Commodities are traded in physical and financial markets. In both cases, transactions can be on a spot or forward basis. Spot transactions involve the immediate, or at least very near term (e.g., next day), transfer of the physical commodity or financial ownership thereof. With forward transactions this transfer occurs at a specified date in the future.

Physical trading eventually leads to the transfer of a commodity from one party to another party. For example, the same lot of natural gas for next day delivery may be purchased and sold several times during a given day, but this lot must be transferred from a seller to a buyer on the delivery date.

Financial markets for commodities, such as the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Intercontinental Exchange (ICE), and the London Metal Exchange (LME) trade various contracts that specify different commodity ownership structures. Futures contracts specify obligations to deliver or receive a given amount of a commodity, at a given price, and at a given maturity. (One month maturity futures typically have the highest volume.) Selling and purchasing, respectively, a futures contract entails the obligation to deliver and receive a commodity. However, most futures traders typically close out their positions via an offsetting trade before maturity. Thus, futures trading need not lead to physical handling of a commodity. Futures prices are set for each delivery date to make the value of the futures contract zero. Hence, there is a single futures price for each delivery date.

Options contracts on futures specify the right to purchase or sell a given futures contracts at a given price and maturity. Call and put options on futures, respectively, given their owners the right to purchase and sell a futures at a contractually specified strike price. The payoff of a call option is the positive part of the difference between the futures price at the option maturity and the strike price. The payoff of a put option is the positive part of the difference between the strike price and the futures price at maturity. Typically, there are options traded with many strikes and expiries.
Additional types of options include calendar spread options on futures contracts. These options give their owners the right to exchange a futures contract with a given maturity for a futures contract with a different maturity at a given strike price. Effectively, these options let investors buy or sell the difference between two futures prices at a given maturity and strike price. The payoff of a call calendar spread option is the maximum between zero and the difference between the spread between the two futures prices and the strike price. The payoff of a put spread option is defined in an analogous manner.

The options described so far are of the European types, meaning that contractually they can be exercised only at maturity. American (respectively, Bermudan) options can be exercised at any time (respectively, a set of predetermined times) before maturity.

Futures, call and put options, and call and put calendar spread option are traded on organized financial exchanges, such as CME, ICE, LME, and NYMEX, which guarantee the clearing of every trade. That is, these trades are insured by the exchange against the risk of counterparty default. These contracts and additional contracts are also traded on over-the-counter (OTC) markets. Trades on OTC markets are directly exposed to greater counterparty default risk.

Physical trading has an important role in the functioning of production, distribution, and consumption processes. The prices of commodities and contracts on commodities are variable and uncertain. They exhibit both seasonality (deterministic variability) and volatility (stochastic variability). Some theories explain the existence of financial (futures) markets based on price-risk aversion/control arguments (see, e.g., Keynes [134], Duffie [80]). Others rely on transaction costs principles (Williams [219, 220]). In the context of this monograph, financial markets play a useful role in terms of the management of commodity conversion assets in the face of variable and uncertain commodity prices, as explained in §§2.3-2.4.

2.3 Conversion Assets and Merchant Operations

Commodities are produced and used for further processing, consumption, and distribution. In this monograph, the industrial facilities that
perform these transformation processes are called commodity conversion assets. Conversion here is used in a broad sense. It refers to

- The production of a commodity, such as the extraction of natural gas from underground wells;
- A physical transformation of a commodity, such as the refining of crude oil, into another commodity, such as gasoline, jet fuel, and naphtha; or
- A change in availability of a commodity, such as the transportation of this commodity in between different locations or its storage over time.

Managing commodity conversion assets thus requires performing one or more operational activities, such as the production, processing, refining, transportation, storage, distribution, and physical trading of commodities.

Consider, for example, a storage asset, such as a grain or metal warehouse, or an underground natural gas storage facility. Storage is used to help match the supply and demand of a given commodity over time. Storage assets feature limited space, and may also exhibit limits on the amount of inventory that can be acquired or disposed of during a given time period. Managing a commodity storage asset requires an inventory trading policy given any space and possible inventory-adjustment (flow) constraints.

As another example, consider a refining asset, such as a crude oil refinery. A crude oil refinery includes various storage tanks for both the inputs, that is, various grades of crude oil, and the intermediate products and outputs. Managing a refinery requires a joint inventory and production policy that determines the inventory levels in these tanks and how the inputs are converted and blended into refined products. In addition, shipping and transportation policies support the sourcing of crude oil and the distribution of refined products.

In practice, the owners of the capacity of a given commodity conversion asset may rent all or part of this capacity to third parties, such as commodity and energy merchants, for given time periods. In particular, this temporary transfer of control of the capacity of a commodity conversion asset occurs when commodity conversion assets are
subject to government regulation. This is the case in the natural gas industry in the United States, where, by federal regulation, the owners of interstate pipelines and storage facilities must make their capacity available to shippers on an open access basis. Tolling agreements play a similar role in the power generation industry whereby physical generator owners can sell off generation revenue to investors. In this monograph, contracts on the capacity of commodity conversion assets are themselves considered commodity conversion assets. In addition to determining an optimal operating policy for such assets, the determination of their market value is an important practical problem, as it forms the basis for contract negotiation.

Commodity conversion assets are embedded in a market environment for the commodities that they convert. Indeed, commodity conversion assets are the building blocks of both physical and financial commodity markets. Commodity production and storage assets, as well as transportation assets, such as pipelines, ships, and related loading and unloading facilities at ports, play a central role in the determination of spot and futures prices (Williams and Wright [221]). Commodity conversion assets can thus be managed taking a merchant perspective, which involves the adjustment of the level of conversion activities to reflect the dynamics of commodity prices. For example, the decision to refine crude oil into refined products depends on the respective spreads between the prices of these products and the price of crude oil. We propose the expression merchants operations to refer to this approach to the management of commodity conversion assets.

The optimal management of merchant operations requires managing operational activities in the face of variable and uncertain commodity prices. The valuation of these operational policies also is important. In general, these are difficult tasks. Interpreting conversion assets as real options facilitates the execution of these tasks.

2.4 Commodity Conversion Assets as Real Options

Managerial flexibility in conversion assets can be viewed as embedded real options on the uncertain evolution of future commodity and energy prices. For example, the ability of oil and natural gas producers
to drill or shut down wells depending on the prevailing oil and natural gas prices can be interpreted as a real option on these prices. Other commodity conversion assets have analogous interpretations as real options on commodity prices. This means that merchant operations of managing commodity conversion assets amounts to optimally exercising specific real options. To make this concept more concrete, consider some examples.

**Simplified commodity production assets and call options.** Consider a natural gas well. Let $c$ be the cost of producing one unit of gas; $s_T$ be the spot price of natural gas at time $T$; and $Q$ the rate of production. For simplicity, ignore the cost of shutting down and resuming production, the time required to do so, and the ability to time the sale of natural gas. The optimal merchant management policy for this simplified natural gas production asset is to produce and sell $Q$ units of natural gas at time $T$ when the spot price of natural gas $s_T$ exceeds the marginal production cost $c$, and do nothing otherwise. The payoff of this policy at time $T$ is

$$\max\{s_T - c, 0\} \cdot Q,$$

which is also the payoff of $Q$ call options on the time $T$ spot price of natural gas $s_T$ with strike price equal to $c$. This observation justifies thinking of the simplified natural gas well at time $T$ as a series of real call options over time on the spot price with strike price equal to the marginal production cost and notional amount equal to the production rate. The notional amount of an option is the “number” of such options owned by a trader. The merchant operations of this asset thus amounts to the optimal exercise of this series of options.

The time $T$ futures price, $F_{T,T}$, for a futures contract with maturity $T$ is identical to the time $T$ spot price, $s_T$. The payoff (2.1) is thus also the payoff of $Q$ call options on the time $T$ futures with maturity $T$ and strike price $c$. As discussed in §2.2, such options are traded on exchanges, such as NYMEX. Thus, the current market value of this simplified commodity production asset, when managed at time $T$ as a merchant operations, can be observed from NYMEX.
It is useful to illustrate the basic principle underlying the theory of option valuation focusing on a European call option on a futures contract. This discussion exemplifies the ideas of option valuation by replication and risk-neutral valuation, which are explained in detail in Chapter 3 and play an important role in merchant operations.

Consider the call option on the time \( T \) futures price with expiry on date \( T \) and strike price equal to \( c \). Assume a binomial process for the futures price dynamics. At time \( T \), the initial time 0 futures price \( F_{0,T} \) can move up to \( u \cdot F_{0,T} \) or down to \( d \cdot F_{0,T} \), with \( 0 < d < 1 < u \). As will become clear shortly, the actual probabilities of the up and down moves are irrelevant. The payoffs of the call option in the up and down states are

\[
\text{call}_{T,u,c} = \max\{u \cdot F_{0,T} - c, 0\}, \\
\text{call}_{T,d,c} = \max\{d \cdot F_{0,T} - c, 0\}.
\]

These payoffs can be replicated by setting up on date 0 a portfolio of \( Y_F \) dollars worth of futures (that is, the futures contract position notional \( n_F \) is \( Y_F/F_{0,T} \)) and \( Y_B \) dollars worth of a risk-free bond. The futures payoffs in the up and down states are

\[
(u \cdot F_{0,T} - F_{0,T})n_F = (u - 1)F_{0,T} \frac{Y_F}{F_{0,T}} = (u - 1)Y_F, \\
(d \cdot F_{0,T} - F_{0,T})n_F = (d - 1)F_{0,T} \frac{Y_F}{F_{0,T}} = (d - 1)Y_F.
\]

The one-month risk-free return \( R \) is \( 1 + r \) (where \( r \) is the one-month risk-free rate), with \( d < R < u \). To replicate, the futures-bond portfolio positions \( Y_F \) and \( Y_B \) are chosen so that

\[
(u - 1)Y_F + R \cdot Y_B = \text{call}_{T,u,c}, \\
(d - 1)Y_F + R \cdot Y_B = \text{call}_{T,d,c}.
\]

The solution to this system of linear equations is

\[
Y_F = \frac{\text{call}_{T,u,c} - \text{call}_{T,d,c}}{u - d}, \\
Y_B = \frac{1}{R} \left[ \left( \frac{1 - d}{u - d} \right) \text{call}_{T,u,c} + \left( \frac{u - 1}{u - d} \right) \text{call}_{T,d,c} \right].
\]
Because a futures contract is worth zero when transacted, the call option value at time 0 is $Y_B$

$$\text{call}_{0,c} = Y_B.$$  

The call option is thus valued by replicating its date $T$ cash flows.

The ratios $(1 - d)/(u - d)$ and $(u - 1)/(u - d)$ are numbers between 0 and 1 and sum to 1. Thus, they can be interpreted as probabilities. In particular, they are known as the risk-neutral probabilities for the up and down states. Label as $q^{RN}$ the up-state risk-neutral probability. Hence, the time 0 option value can be written as the expected value of the time $T$ option payoffs evaluated using the risk-neutral probabilities $q^{RN} \equiv (1 - d)/(u - d)$ and $1 - q^{RN} \equiv (u - 1)/(u - d)$, discounted at the risk-free discount factor $1/R$:

$$\text{call}_{0,c} = \frac{1}{R} \left[ q^{RN} \cdot \text{call}_{T,u,c} + (1 - q^{RN}) \cdot \text{call}_{T,d,c} \right].$$

In words, valuation by replication reduces to the computation of a mathematical expectation discounted using the risk-free discount factor. Because this expectation depends on the risk-neutral, rather than the actual, probabilities of reaching the up or down states, this valuation approach is known as risk-neutral valuation. That is, for valuation (and policy optimization) purposes we can focus on the evolution of the futures price in an alternate risk-neutral world in which no adjustment for risk is required when discounting.

Risk-neutral valuation is a general approach. It is not specific to this simplified setting. This example can be substantially generalized by (i) subdividing the time period between dates 0 and $T$ into small time intervals each of length $\Delta t$; (ii) assuming that during each such time interval the futures price with date $t$ delivery at the start of this interval can increase by a proportional factor $u$ equal to $\exp(\sigma \sqrt{\Delta t})$ or decrease by a factor $d$ equal to $\exp(-\sigma \sqrt{\Delta t})$, where $\sigma$ is known as the volatility parameter; and (iii) letting the length of these time intervals become smaller and smaller. In the limit, the risk-neutral dynamics of the futures price $F_{t,T}$ converge to Geometric Brownian Motion, with the stochastic differential equation

$$\frac{dF_{t,T}}{F_{t,T}} = \sigma dZ_t,$$
where $dF_{t,T}$ is the instantaneous change in the futures price $F_{t,T}$ and $dZ_t$ is a Standard Brownian Motion increment. That is, $dZ_t$ is normally distributed with mean 0 and standard deviation $dT$ (the length of an instantaneous time interval). The resulting risk-neutral probability distribution of the time $T$ futures price on date $T$, $F_{T,T}$, conditional on the information available on date 0, $F_{0,T}$, is lognormal with mean $F_{0,T}$ and variance $F_{0,T}^2[\exp(\sigma^2T) - 1]$.1 Letting $E_{0}^{RN}$ be risk-neutral expectation and using continuous discounting, the value of the call option on date 0 is

$$\text{call}_{0,c} = e^{-rT}E_{0}^{RN}\left[\max\{F_{T,T} - c, 0\}\right],$$

which, given that the futures price follows a Geometric Brownian Motion, leads to the valuation formula

$$\text{call}_{0,c} = e^{-rT} [F_{0,T} \cdot \Phi(d_1) - c \cdot \Phi(d_2)], \quad (2.2)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, $d_1$ is $[\ln(F_{0,T}/c) + \sigma^2T/2]/(\sigma\sqrt{T})$, and $d_2$ is $d_1 - \sigma\sqrt{T}$.

**Transportation assets and spread options.** Consider a pipeline that transports natural gas from Houston, Texas, to New York City, New York. Suppose that a merchant rents an amount $Q$ of the capacity of this pipeline for a time period, say one month starting at time $T$. This merchant can maximize the market value of this monthly block of capacity by optimally trading natural gas on the Houston and New York City spot markets for natural gas. At time $T$ the merchant optimally purchases an amount of natural gas equal to the rented pipeline capacity on the Houston spot market, ships this natural to New York City using this capacity, and sells the shipped natural gas on the New York City spot market if the New York City natural gas price exceeds the Houston natural gas price net of the shipping cost.

Let $s_{1,T}$ and $s_{2,T}$ be the spot prices at time $T$ for the Houston and New York City markets. Denote by $c$ the marginal shipping cost. The time $T$ payoff of the optimal shipping policy is

$$\max\{s_{2,T} - s_{1,T} - c, 0\} \cdot Q. \quad (2.3)$$

1Equivalently, the resulting risk-neutral distribution of $\ln(F_{T,T})$ is normal with mean $\ln(F_{0,T}) - \sigma^2T/2$ and variance $\sigma^2T$. 


This payoff recognizes that natural gas is simultaneously injected into the pipeline in Houston and delivered in New York City at time $T$. This simultaneous receipt and delivery of natural gas is realistic because natural gas is moved by displacement and the entire pipeline from Houston to New York City is filled up with natural gas. It is clear, however, that the delivered natural gas is not physically the same injected natural gas. For simplicity, the payoff (2.3) ignores any fuel used by the pipeline compressors to ship natural gas. It also assumes trading of one monthly block of natural gas, rather than daily trading within the month.

The payoff (2.3) corresponds to the payoff of $Q$ call spread options on the spot prices $s_{1,T}$ and $s_{2,T}$ with strike price $c$. Different from the payoff (2.1) for the simplified natural gas production asset, the payoff (2.3) involves two distinct commodities: natural gas in Houston at time $T$ and natural gas in New York City at time $T$. A contract on natural gas pipeline capacity is thus a cross-commodity conversion asset, which can be interpreted as a real call option on the time $T$ spread between the spot prices in Houston and New York City with strike price equal to the marginal shipping cost and notional amount equal to the rented pipeline capacity. The merchant operations of this asset corresponds to the optimal exercise of this option.

Other transportation assets share this spread option feature with natural gas pipelines. Examples include power lines, as well as crude oil tankers and trains that haul coal rail cars. The difference between the latter assets and power lines and natural gas pipelines is that tanker and train payoffs are defined on commodity prices at both multiple location and dates, due to the longer transportation lead time. Transportation assets can also entail multiple sourcing and delivery locations. This aspect can be characterized as a rainbow option, that is, an option to choose the maximum or minimum among multiple prices.

Exchanges such as NYMEX and ICE trade basis swaps for several locations in the United States. These contracts are essentially futures contracts on the difference between the futures price at a given location and the futures price at Henry Hub, the delivery location for the NYMEX natural gas futures contract. The time $T$ spot prices for Houston and New York City, $s_{1,T}$ and $s_{2,T}$, should be identical to the
slopes of the time $T$ Henry Hub futures price and Houston and New York City basis swaps’ prices, respectively, both with time $T$ maturity. Denote these sums by $F_{1,T,T}$ and $F_{2,T,T}$. Thus, the payoff (2.3) is that of a spread option on the difference between the futures prices $F_{2,T,T}$ and $F_{1,T,T}$ net of the strike price $c$.

Cross-commodity spread options are not directly traded on NYMEX or ICE but they may be traded on OTC markets. However, the time 0 < $T$ market value spread $0_{T,c}$ of each one of these options can be determined using risk-neutral valuation, based on a stochastic model of the joint risk-neutral evolution of the prices $F_{1,t,T}$ and $F_{2,t,T}$ during the time interval $[0,T]$:

$$\text{spread}_{0,c} = e^{-rT} \mathbb{E}^{RN} \left[ \max \{ F_{2,T,T} - F_{1,T,T} - c, 0 \} \right].$$

For example, suppose that the futures prices $F_{1,t,T}$ and $F_{2,t,T}$ evolve as correlated Geometric Brownian Motions in the risk-neutral world:

$$\frac{dF_{1,t,T}}{F_{1,t,T}} = \sigma_1 dZ_{1,t},$$
$$\frac{dF_{2,t,T}}{F_{2,t,T}} = \sigma_2 dZ_{2,t},$$
$$dZ_{1,t} dZ_{2,t} = \rho dt,$$

where $\sigma_1$ and $\sigma_2$ are the volatilities of the two futures prices, and $dZ_{1,t}$ and $dZ_{2,t}$ are Standard Brownian Motion increments with instantaneous correlation equal to $\rho$. In this case a closed-form formula for the spread option price $\text{spread}_{0,c}$ is not available, but this price can be approximated as

$$\text{spread}_{0,c} \approx e^{-rT} [F_{2,0,T} \Phi(d_2) - (F_{1,0,T} + c) \Phi(d_1)], \quad (2.4)$$

where

$$d_2 = \frac{\ln(F_{2,0,T}/(F_{1,0,T} + c)) + \sigma_{1,2}^2 T/2}{\sigma_{1,2} \sqrt{T}},$$
$$d_1 = d_2 - \sigma_{1,2} \sqrt{T},$$
$$\sigma_{1,2} = \sqrt{\sigma_2^2 - 2 \rho \sigma_1 \sigma_2 (F_{1,0,T}/(F_{1,0,T} + c) + \left( \frac{\sigma_1 F_{1,0,T}}{F_{1,0,T} + c} \right)^2).}$$


### Simplified consumption assets and put options.

Commodity consumption assets play the opposite role of commodity production assets. Consider a firm that employs a given commodity as its major input to its manufacturing process. For example, the input commodity could be copper for a company that manufactures pipes. For simplicity, suppose that the firm can sell its production at time $T$ at the fixed price $K$ (this is not entirely realistic, as the firm could adjust its pricing policy according to the input price, but represents a situation where output prices are fixed in the short term). This price is net of other (nonrandom) variable manufacturing costs. Suppose that production is known with certainty and is equal to $Q$. Assume, also for simplicity, that the firm does not carry inventory of the input commodity.

The manufacturer’s optimal production policy is to purchase the input commodity, produce, and sell its output at time $T$ if the output price exceeds the input price. The payoff of this policy is

$$\max\{K - s_T, 0\} \cdot Q. \quad (2.5)$$

The payoff (2.5) is also the payoff of $Q$ European put options on the spot input price at time $T$ with strike price equal to $K$. This observation justifies interpreting the commodity consumption asset as a real put option on the time $T$ spot price of the commodity with strike price equal to the output price and notional equal to the manufacturing capacity. The merchant operations of the manufacturing asset is identical to the optimal exercise of this put option. The time $0 < T$ market value of the manufacturing asset managed as a merchant operations is thus the time 0 value of this futures option. Similar to European call options on commodity futures prices, European put options on such prices are also traded on exchanges (such as LME). The time 0 value of the commodity conversion asset when managed as a merchant operations at time $T$ can thus be observed in the market. It is the value of the futures put option multiplied by the production capacity for time $T$. Moreover, risk-neutral valuation can be applied to the valuation of each of these options. Indeed, the time 0 value of a futures put option with expiry on date $T$ and strike price $K$ can be obtained from the time 0 value of the futures call option with the same maturity $T$ and strike price $K$ as

$$\text{put}_{0,K} = \text{call}_{0,K} + e^{-rT}(K - F_{0,T}). \quad (2.6)$$
These examples illustrate that interpreting commodity conversion assets as real options and the merchant operations of such assets as the optimal exercise of specific real options is useful because it leads to the maximization of the *market values* of these assets.

### 2.5 Outline of this Monograph

In some cases, managing a commodity conversion asset as merchant operations is simple, such as in the examples discussed in §2.4. However, for most commodity conversion assets, doing this requires the development of mathematical models that maximize the market value of the merchant operating policy based on a stochastic representation of the evolution of commodity and energy prices, subject to certain market pricing consistency criteria. Specifically, the merchant operating policy prescribes how to optimally exercise the managerial flexibility embedded in a commodity conversion asset, that is, the real options that this asset represents, and the market pricing consistency criteria ensure that this policy maximizes the market value of this asset.

In theory, the development and use of such real option models requires the existence of frictionless financial markets for commodity contracts. Some commodity and energy industries come close to satisfying these requirements, e.g., grains, soybean, natural gas, and oil all have fairly liquid financial markets (e.g., CME, ICE, NYMEX). The real option modeling approach remains a useful approximation even when these requirements are not perfectly satisfied, as it provides a consistent framework for devising operating policies that strive to maximize the market value of an asset. Chapter 3 outlines the theory behind the valuation of real options. Chapter 4 illustrates a number of widely used models of the evolution of commodity and energy prices.

The transportation assets discussed in §2.4 are fairly realistic representations of how these assets are operated in practice. In contrast, the commodity production and consumption assets presented in §2.4 simplify many important aspects of how such assets are managed. In particular, these simplified examples neglect a basic aspect that distinguishes most storable commodity conversion assets: *inventory*. For commodity production assets, inventory represents the reserve of com-
modity that can be produced and sold over time. For commodity consumption assets, inventory is the input stored in tanks or stockpiles that is available for use in the manufacturing process. Refining assets feature multiple types of inventory, both for inputs and outputs, but also, possibly, for intermediate products. Thus, for refining, storage is bundled with cross-commodity transformations. Inventory is important because it links decisions across multiple dates and can lead to complex optimal quantity decisions, that is, decisions about a commodity conversion asset operating scale. Optimal merchant operations of commodity storage thus extends the optimal valuation and management of American and Bermudan options with their one-time binary exercise. Switching costs incurred to alter the scale of operations of a commodity conversion asset can also cause intertemporal linkages across decisions.

From this perspective, storage is a foundational element of commodity conversion assets. In a simplified model, a single commodity is purchased from the market at the prevailing spot price, held in stock at a warehouse, and resold back to the spot market at a future date. Figure 2.1 illustrates the essential dynamics of commodity storage. Chapters 5-6 deal with the structure of the optimal policy for the commodity storage asset and the benchmarking of policies used to manage such assets in practice, respectively. Chapter 7 shows that the optimal policy structure for a commodity storage asset remains relevant for the
merchant operations of other assets, including inventory disposal and acquisition assets and swing assets. In contrast, realistic real option modeling of more complicated commodity conversion assets remains an open area of research and applications. This is one of the trends discussed in Chapter 8.

Figure 2.2 summarizes the merchant operations framework and how the chapters of this monograph relate to the components of this framework. Physical markets include the spot and forward markets where the exchange of commodities and the conversions of these commodities from inputs to outputs occur. Conversion assets are interpreted as real options on the input and/or output commodity prices. Financial markets trade various contracts on these input and output commodities. Physical and financial markets are linked: (i) financial markets provide the information needed to devise operating policies for commodity conversion assets that (strive to) maximize the market-value of these assets, and (ii) the current and future supply and demand conditions in physical markets are reflected in the market prices of the contracts traded in the financial markets. Chapter 3 deals with the role played by financial markets in the valuation of the conversion asset physical cash flows. Chapter 4 presents various models of the evolution of spot
and futures prices and the impact of supply and demand conditions on price dynamics. Chapters 5-7 focus on operating policies to manage various commodity conversion assets. Chapter 8 discusses directions for future research that touch on all the components of merchant operations, including the impact of physical markets on financial markets.

This monograph takes an integrated finance and operations perspective on commodity and energy merchant operations. However, Chapters 3-4 are more finance oriented while Chapters 5-7 are more operations oriented. Those readers who are more interested in finance can focus on Chapters 3-4 and skim Chapters 5-7, while readers who are more interested in operations can proceed to Chapters 5-7 after skimming Chapters 3-4.

2.6 Notes

The categorization of commodity groups and types in §2.1 is from Geman [99, Table 1, p. 13]. Geman [99] discusses in detail the markets for the commodity groups introduced in §2.1. Schofield [181] provides a detailed account of commodity markets and derivatives. Clewlow and Strickland [59], Eydeland and Wolyniec [84], Geman [99], Fiorenzani [91], Burger et al. [39], Pilipovic [167], and Fiorenzani et al. [92] illustrate various contracts traded in commodity financial markets, as well as real options models for valuation and risk management of energy derivatives and assets. Hull [119, p. 587] is a comprehensive introduction to derivatives contracts, including futures and options. The collection edited by Ronn [175] includes several contributions on the use of real options in energy management. Leppard [145] provides a non-technical introduction to energy derivatives and their role in the risk management of energy assets.

The real option literature is vast, and no attempt is made here to provide a comprehensive coverage. Dixit and Pindyck [76] and Tri-georgis [211] provide introductions to real option concepts. Clewlow and Strickland [59], Ronn [175], Eydeland and Wolyniec [84], Kamiński [126], Geman [99], Hull [119, Chapter 33], and Luenberger [151] present various commodity and/or energy applications. Smith and Mccardle [197] discuss practical applications of real option concepts in the
Commodity Conversion Assets and Merchant Operations

energy industry. Benth et al. [12] is an advanced treatment of energy price evolution models, a topic discussed in Chapter 4.

The link between the management of commodity conversion assets as merchant operations and the management of real options is implicit in most of this literature. In practice, the merchant operations approach to managing commodity conversion assets has been embraced by commercial banks (see, e.g., Davis [64]) and energy merchants (Ronn [175], Kaminski [126]).

The discussion of the commodity conversion assets considered in §2.4 is based on the work of Brennan and Schwartz [35] on the real option valuation of natural resource production, the research of Deng et al. [72], Eydeland and Wolyniec [84, pp. 59-62], Fleten et al. [93], and Secomandi [185] on the real option valuation of electricity and natural gas transportation infrastructure, and the paper by McDonald and Siegel [156] on the real option valuation of manufacturing firms. Secomandi and Wang [191] consider natural gas transport assets with a network structure.

The illustration in §2.4 of the valuation of a call option on a futures contract using the binomial model follows Luenberger [151, pages 383-386]. Cox et al. [62] present the binomial model for the valuation of options on a stock and its convergence to the model of Black and Scholes [20], in which the stock price dynamics follow a Geometric Brownian Motion. Formula (2.2) is the Black [19] formula for the price of a call option on a futures contract (see Hull [119, p. 370] for a textbook discussion of this formula). Formula (2.4) is known as Kirk’s approximation for the price of a futures spread option (see Carmona and Durrleman [44]). Expression (2.6) is known as put-call parity (see Hull [119, p. 365]).

Sick [194] discusses the market consistency criteria mentioned at the beginning of §2.5. In particular, Sick [194] points out that the optimal management of commodity and energy real options often gives rise to stochastic dynamic programs whose formulation must satisfy these criteria. Such market consistency criteria are based on assumptions that can be restrictive in applications. Smith [195] points out that even when these assumptions are not fully satisfied, enforcing these
criteria provides a consistent framework for informing and supporting managerial decision-making.
This chapter presents a framework for pricing future cash flows paid off by real and financial commodity options. Section 3.1 introduces the idea of risk neutral valuation and its connection to option valuation which is consistent with the market prices of related traded securities. Section 3.2 explains risk neutral valuation in statically complete markets. Section 3.3 describes risk neutral valuation in dynamically complete markets based on Black and Scholes [20] and Black [19]. Section 3.4 explains the role of price-of-risk assumptions in risk neutral valuation in incomplete markets. Section 3.5 discusses model calibration. Section 3.6 summarizes. Section 3.7 gives pointers to the literature.

3.1 Introduction

Option valuation depends, in general, on the probability beliefs and preferences of investors trading in the market and on the option payoff function. For commodity options, the option payoffs are contingent on commodity prices. The evolution of commodity prices can be represented formally as functions on a probability space \((\Omega, \mathcal{F}, P)\) where \(\Omega\) is the set of all possible states, \(\mathcal{F} = \{\mathcal{F}_t\}\) is a filtration which de-
scribes the possible evolution over time of information about the realized state $\omega \in \Omega$, and $P$ is a probability measure over informational states $\omega_t \in \mathcal{F}_t$. To be more concrete, a state $\omega$ here is an omniscient description of supply and demand conditions for a commodity at all dates, while an informational state $\omega_t$ can be thought of as what has been publicly reported about commodity market conditions in the *Wall Street Journal* (or the financial press more generally) up through a given date $t$. A probability space like this generalizes the simple binomial tree in Chapter 2 to allow for continuous time, continuous states, and multiple state variables.

Given a probability space, let $c^*_T$ denote an option payoff at a future expiration date $T$ where $c^*_T$ must be measurable with respect to $\mathcal{F}_T$. In other words, $c^*_T$ can only depend on commodity price information that will be known at date $T$. The payoff $c^*_T$ can be determined contractually (as with exchange-traded and OTC call and put options) or operationally (as with conversion assets with embedded real options). Moreover, the operating policy for conversion assets must be both feasible given physical engineering constraints and measurable with respect to the information available over time given the filtration $\mathcal{F}$.

In the absence of arbitrage, a non-negative function $\varphi(\omega_T|\omega_t)$ exists such that any measurable payoff function $c^*_T$ can be valued at date $t<T$ as

$$c_t = \int_{\omega_T \in \mathcal{F}_T} c^*_T(\omega_T) \varphi(\omega_T|\omega_t) d\omega_T$$

(3.1)

where $\omega_T$ is a possible future state given the information $\mathcal{F}_T$ at date $T$, $\omega_t$ is the state given current information $\mathcal{F}_t$ at date $t$, and $\varphi(\omega_T|\omega_t)$ is a *state price density* giving the market value in state $\omega_t$ at date $t$ of a state-contingent dollar in state $\omega_T$ at date $T$. The state prices incorporate everything that matters at date $t$ for valuing future $\omega_T$-contingent cash. In particular, state prices depend on the time-value-of-money between dates $t$ and $T$, the market’s beliefs about the probability of state $\omega_T$ occurring at $T$ given that $\omega_t$ is the current state at $t$, and the market’s preferences in state $\omega_t$ for state-contingent cash in the future state $\omega_T$. For example, the market may have a preference for

---

1See Harrison and Kreps [113] and Duffie [81] chapters 1 and 6 for more on multi-period asset pricing and state prices.
insurance in some futures states (e.g., down-market states in a CAPM world) but not in others (e.g., up-market states). If riskless interest rates are a constant \( r \), the state price valuation equation (3.1) can be manipulated as

\[
ct = \left[ \int_{\omega_T \in F_T} c_T^*(\omega_T) \frac{\varphi(\omega_T|\omega_t)}{\int_{\omega_T \in F_T} \varphi(\omega_T|\omega_t) d\omega_T} d\omega_T \right] \left[ \int_{\omega_T \in F_T} \varphi(\omega_T|\omega_t) d\omega_T \right]
\]

\[
= \left[ \int_{\omega_T \in F_T} c_T^*(\omega_T) q^{RN}(\omega_T|\omega_t) d\omega_T \right] e^{-r(T-t)}
\]

(3.2)

to obtain the risk neutral (RN) valuation equation\(^2\)

\[
ct = E_t^{RN}[c_T^*] e^{-r(T-t)}
\]

(3.3)

The second equality in (3.2) follows from (i) the fact that the integral of the state prices \( \int_{\omega_T \in F_T} \varphi(\omega_T|\omega_t) d\omega_T \) equals the price of a riskless discount bond \( e^{-r(T-t)} \) (i.e., of a riskless payoff of $1 in each possible state \( \omega_T \) at \( T \)) and (ii) an interpretation of scaled state prices as preference-adjusted risk neutral (RN) probabilities \( q^{RN}(\omega_T|\omega_t) \). The RN probabilities are non-negative and integrate to 1 but differ from the objective probabilities because the RN probabilities are constructed from state prices which depend on market preferences as well as on objective probabilities.\(^3\)

The RN valuation equation (3.3) has important theoretical and practical implications. One immediate implication is that all of the properties of option prices and their risk characteristics follow directly from the option payoff formula and the RN dynamics of the state variables driving the commodity price filtration. Another implication of (3.3) is that the annualized RN expected return on any claim on future state-contingent cash flows – that is to say, on any investible asset – is the risk-free interest rate \( r \).\(^4\) However, it should be noted that the RN

\(^2\)See Schwartz [182], Miltersen and Schwartz [158] and Casassus and Collin-Dufresne [47] for models of commodity option pricing with stochastic interest rates.

\(^3\)Sometimes RN probabilities are decomposed as \( q^{RN}(\omega_T|\omega_t) = p(\omega_T|\omega_t) m(\omega_T|\omega_t) \) where \( p(\omega_T|\omega_t) \) is the objective probability and \( m(\omega_T|\omega_t) \) is called the pricing kernel and represents the equilibrium pricing impact of investor preferences.

\(^4\)To see this, note that the return on an asset \( c_t \) over a time interval \( \Delta t \) is

\[
E_t^{RN} [c_t^*] e^{-r(T-t-\Delta t)} - E_t^{RN} [c_t^*] e^{-r(T-t-\Delta t)}/c_t.
\]

Using iterated expectations under the RN measure and then taking the limit as \( \Delta t \to 0 \) gives \( E_t^{RN} [dc_t/c_t] = r dt \).
valuation representation of asset prices imposes no restrictions on the RN drift of variables which are not prices of investible assets. This point will be important in our discussion of the RN dynamics of non-asset commodity prices (e.g., commodities that are perishable or which are not practical stores of value like electricity or some agricultural goods) as well as for the RN dynamics of variable like weather and stochastic volatility factors.

A practical use of RN valuation is that equation (3.3) provides a tractable numerical procedure for computing the value of options when analytic formulas for option prices are not available. First, N realizations of the underlying state $\omega_T$ are simulated under the RN probability measure using Monte Carlo. Second, option payoffs $c_T^*(i)$ are computed for each simulated realization $i = 1, \ldots, N$ of the state $\omega_T(i)$, averaged, and discounted to get an estimated option valuation:

$$\hat{c}_t = \frac{\sum_{i=1}^N c_T^*(i)}{N} e^{-r(T-t)}.$$

(3.4)

Since averages are unbiased estimators of expected values, the Monte Carlo estimate $\hat{c}_t$ is an unbiased estimate of the option price $c_t$. This Monte Carlo implementation of RN valuation is a workhorse tool for valuing gas storage, power plants, and other complex commodity conversion assets.

Real options often have cash flows, not at just a single date, but rather a stream of cash flows $c_1^*, c_2^*, \ldots, c_M^*$ at a sequence of dates $T_1, T_2, \ldots, T_M$. Using RN valuation, the value of a sum of cash flows is

$$c_t = \sum_{i=1}^M \mathbb{E}_t^{RN}[c_{T_i}^*] e^{-r(T_i-t)}.$$

(3.5)

As discussed in Chapter 2, the stream of cash flows for conversion assets are a stream of operating cash flows over time. For example, mines generate streams of net profits over time tied to mineral prices. Electric power plants produce streams of operating profits tied to the spark spread between power and fuel prices. Natural gas storage – which is considered in detail in Chapters 5 and 6 – yields a stream of negative cash flows (in months in which gas is purchased and stored) and positive cash flows (in months in which stored gas is withdrawn and sold).
The purpose of an option pricing model is to specify the RN probabilities with which to value future state-contingent cash flows. Since the existence of state prices – and, hence, of RN probabilities – is ensured by absence of arbitrage, the focus in this chapter is on two related issues: When are RN probabilities unique? And how are RN probabilities identified? In particular, this chapter reviews three approaches to identify RN probabilities. The first approach is possible in a statically complete market. The second is possible in a dynamically complete market. The third requires price-of-risk assumptions when the market is incomplete. While the discussion here is exposited in terms of commodities, the uniqueness and identification of the RN dynamics of an option’s underlying variable are generic issues. They arise with options on any investible asset (e.g., stocks, bonds) and on underlyings which are not directly investible (e.g., interest rates, weather, volatility, default events).

A key consideration in option pricing is what exactly constitute a “state.” In its most complete form, an informational state $\omega_t \in F_t$ in a commodity option pricing model includes the current spot price $s_t$ and also any other factors (i.e., other random variables like stochastic volatility) that affect the future dynamics of spot prices plus the prior history of all state variables. For example, weather would be a natural state variable for electricity prices and international political uncertainty might be a state variable for oil price volatility. In this chapter, the focus will be on state variable dynamics and option pricing from a fairly generic vantage point. Chapter 4 then describes a variety of different commodity option pricing models in terms of their specific representations of states, price dynamics, and RN probabilities.

### 3.2 Statically Complete Markets

Traded securities are bundles of state-contingent cash flows in various future dates and future states. A market is said to be statically complete if any combination of future state-contingent payoffs can be constructed using buy-and-hold positions in market-traded securities. More formally, a market is statically complete if there are as many traded securities with linearly independent cash flows as there are
3.2. Statically Complete Markets

In a statically complete market, state prices can be recovered from the market prices of traded securities. Let \( t_0 \) denote the current date and let \( t_1, \ldots, t_M \) denote discrete future payment dates. Let \( K_j \) denote the total number of discrete (for simplicity) possible states at future payment date \( t_j \). The total number of future date-state pairs is \( K = \sum_{j=1}^{M} K_j \). Let \( \varphi_k \) denote the state price at date \( t_0 \) of a state-contingent dollar in the \( k \)th possible future state-date pair. Let \( \text{CF}_{i,k} \) denote the future cash flow paid off by a particular traded security \( i \) in a possible future date-state \( k \) and let \( P_{i,t_0} \) denote the price at date \( t_0 \) of security \( i \). State prices equate the values of the future security cash flows with the current date \( t_0 \) security prices. If we have \( N \) traded securities with linearly independent future cash flows, then the market prices of these securities imply the system of linear equations

\[
P_{1,t_0} = \varphi_1 \cdot \text{CF}_{1,1} + \varphi_2 \cdot \text{CF}_{1,2} + \cdots + \varphi_K \cdot \text{CF}_{1,K},
\]
\[
P_{2,t_0} = \varphi_1 \cdot \text{CF}_{2,1} + \varphi_2 \cdot \text{CF}_{2,2} + \cdots + \varphi_K \cdot \text{CF}_{2,K},
\]
\[\vdots\]
\[
P_{N,t_0} = \varphi_1 \cdot \text{CF}_{N,1} + \varphi_2 \cdot \text{CF}_{N,2} + \cdots + \varphi_K \cdot \text{CF}_{N,K}.
\]

The solution for the \( \varphi_k \)s in (3.6) is unique if the number of traded securities with linearly independent future cash flows, \( N \), equals the number of future dates-states \( K \). However, in practice, there are typically many more future dates-states than traded securities, so markets are rarely statically complete. Hence, state prices cannot be uniquely identified algebraically from an under-determined system of \( N \) equations in \( K > N \) unknowns. The problem becomes even more severe with continuous-states (as in (3.2)) where there are an infinite number of possible states and only a finite number of traded securities.

State prices can still exist in incomplete markets, but they are not uniquely identified by traded security prices. In the absence of arbitrage, there are multiple possible state price vectors which are each consistent with the traded security prices in the market. Option pricing models impose additional structure on state prices in order to determine state prices in an incomplete market. An option pricing model is correct if its assumed additional structure on state prices is consis-
tent with the structure in fact imposed on state prices by the actual economics of the market

\section*{3.3 Dynamically Complete Markets}

Markets which are statically incomplete may still be \textit{dynamically complete} if there are dynamic trading strategies with traded securities which can replicate any state-contingent payoff. In particular, a dynamic trading strategy is a rule for buying and selling securities over time. The link between dynamic completeness and option pricing was first recognized in Black and Scholes \cite{20}. There are two approaches to commodity option pricing in dynamically complete markets. The first assumes that the underlying commodity is itself a traded asset. This approach applies to commodities like gold which are investible stores of value over time. The second approach, due to Black \cite{19}, assumes that a traded commodity futures contract is available. This second approach can be used to price a restricted set of option payoffs when the underlying physical commodity itself is not an investible asset. In the context of physical conversion assets, for example, it can be used to price real options on electricity (e.g., power plants) when electricity futures are traded but electricity itself is not directly investible.

Most of the option pricing models used in practice have continuous time and continuous states. This raises the mathematical bar. Unlike the binomial tree in Chapter 2 and the finite $K$ states in Section 3.2, stochastic calculus is used to model the underlying variable dynamics in terms of stochastic differential equations.

\textbf{Commodities which are traded assets:} The Black-Scholes approach to identify RN dynamics follows from two key assumptions. First, the underlying commodity is assumed to be an investible asset (e.g., like gold). Second, the objective dynamics of the spot price of the commodity $s_t$ are assumed to follow a general Ito process:\footnote{See Shreve \cite{193} for more on Ito processes.}

$$ds_t = \mu(s_t, t)s_t dt + v(s_t, t)s_t dZ_t.$$  \hspace{1cm} (3.7)

where $dZ_t$ is a Standard Brownian Motion. These Ito dynamics gen-
eralize the original Black and Scholes \[20\] derivation for Geometric Brownian Motion prices where the local drifts and volatilities are constants \( \mu(s_t, t) = \mu \) and \( \sigma(s_t, t) = \sigma \). The assumption of an univariate Ito process for spot prices means that the spot price \( s_t \) itself is the only state variable needed to describe commodity price dynamics. In particular, there are no additional factors (e.g., weather) affecting its drift and volatility. Given these two assumptions, generalized Black-Scholes RN dynamics can be derived in two steps.

The first step is the derivation of the Black-Scholes partial differential equation. Over time only two things change in a Black-Scholes economy – the spot price \( s_t \) and the date \( t \). As a result, option prices are functions \( c(s_t, t) \) of just two state variables, \( s_t \) and \( t \), where everything else at date \( t \) is parametrically fixed by assumption (e.g., interest rates, the forms of the local drift and volatility functions \( \mu(\cdot, \cdot) \) and \( \sigma(\cdot, \cdot) \)), contractually fixed (e.g., option strike prices and expiration dates), and historically fixed (e.g., the path of prior spot prices before date \( t \) when valuing path-dependent options). The valuation function \( c(s_t, t) \) must, however, be consistent with absence of arbitrage. This no-arbitrage condition imposes restrictions on the types of functions \( c \) which value commodity price-contingent future cash flows.

Consider a portfolio consisting of a long position in a given option combined with a position consisting of \( -\delta \) units of the underlying commodity. In particular, this is only possible if the physical commodity is a non-perishable, storable, and, hence, investible asset that can be included in an investment portfolio. At date \( t \) this portfolio is worth \( c_t - \delta s_t \). Since \( c_t \) is a function \( c(s_t, t) \), Ito’s Lemma tells us how this portfolio’s value changes in response to the passage of time \( dt \) and changes \( ds_t \) in the underlying commodity spot price:\(^6\)

\[
dc_t - \delta ds_t = \left[ \left( \frac{\partial c}{\partial t} + \frac{1}{2} \sigma^2(s_t, t) s_t^2 \frac{\partial^2 c}{\partial s^2}\right) dt + \frac{\partial c}{\partial s} ds_t \right] - \delta ds_t \quad (3.8)
\]

provided that the partial derivatives \( \partial c/\partial t \), \( \partial c/\partial s \), and \( \partial^2 c/\partial s^2 \) exist and are continuous.

\(^6\) The function \( c(s_t, t) \) is parameterized by everything that is fixed at date \( t \) including process parameters, contractual payoff terms, and (for path-dependent options) the history of past spot prices. For an explanation of Ito’s Lemma, see Shreve [193] or Hull [119].
The portfolio value dynamics have two components: A deterministic part due to the nonrandom passage of time $[\partial c/\partial t + (1/2)v^2(s_t, t)s_t^2\partial^2 c/\partial s^2] \, dt$ and a risky part due to random changes in the spot price $(\partial c/\partial s - \delta) \, ds_t$. Setting the commodity position $\delta = \partial c/\partial s$ makes the combined option/commodity portfolio riskless. In particular, the implicit exposure to spot price risk through the option, $\partial c/\partial s$, is offset by the direct exposure from the short $\delta = \partial c/\partial s$ commodity position. To avoid arbitrage between this riskless option/commodity portfolio and riskless t-bills, the function $c(s_t, t)$ must satisfy

$$dc_t - \frac{\partial c}{\partial s} ds_t = \left(c_t - \frac{\partial c}{\partial s}s_t\right) r \, dt.$$  

(3.9)

In words, the change in the value of the hedged option-commodity portfolio must be the same as would be earned if an equal amount of money, $c_t - (\partial c/\partial s)s_t$, were instead invested at the riskless rate. This leads to the Black-Scholes partial differential equation (PDE)

$$\frac{\partial c}{\partial t} + \frac{1}{2}v^2(s_t, t)s_t^2\frac{\partial^2 c}{\partial s^2} = \left(c_t - \frac{\partial c}{\partial s}s_t\right) r.$$  

(3.10)

This equation has infinitely many solutions $c$ corresponding to different arbitrage-free valuation functions. To pick out the specific function which prices a given option, a boundary condition is imposed on (3.10) requiring that the function $c(s_T, T)$ equals the payoff function $c^*_T$ on the payoff date $T$.

In the special case of a Geometric Brownian Motion, for which the local volatility is constant, $v(s_t, t) = \sigma$, the Black-Scholes PDE can be explicitly solved to value a variety of options (including European calls and puts) in closed-form. However, our interest here is not closed-form solutions per se, but rather to identify RN commodity price dynamics as an input into RN valuation for a large class of heteroskedastic processes.

As an aside, note that the payoffs of any commodity option can be exactly replicated, under the Black-Scholes assumptions, by a self-financing dynamic trading strategy. Rearranging (3.9) gives

$$dc_t = \frac{\partial c}{\partial s} ds_t + \left(c_t - \frac{\partial c}{\partial s}s_t\right) r dt$$  

(3.11)
which says that the dollar price change in an option $dc_t$ over each instant $dt$ is exactly mirrored by the dollar change in the value of a portfolio consisting of $\partial c/\partial s$ units of the commodity and $c_t - (\partial c/\partial s)s_t$ dollars in riskless bonds.\footnote{The cost of the replicating stock-bond portfolio on day $t$ is the price of the option $\partial c/\partial s s_t + c_t - \partial c/\partial s s_t = c_t$. Thus, the replicating trading strategy is "self-financing" through time because the change in the value of the portfolio exactly matches the change in the cost of the replicating portfolio instant by instant.} This argument is the essential intuition for showing that the Black-Scholes market is dynamically complete.\footnote{As presented, the argument here only applies to option payoff functions $c^*_T$ which are limits of functions $c(s_t, t)$ with continuous partial derivatives $\partial c/\partial t$, $\partial c/\partial s$, and $\partial^2 c/\partial s^2$ as $t \to T$. See Shreve [193] for a proof that a replicating strategy exists in general for any measurable payoff function $c^*_T$ of $\omega_T$.} A discretized version of (3.11) is often used in practice as an approximate hedge for option positions. In particular, taking an offsetting position of $\partial c/\partial s$ units of the underlying investible commodity to hedge commodity price risk in an option position – with or without the t-bill position – is known as $\text{delta hedging}$.}

The second step of the Black-Scholes identification of the RN commodity price dynamics uses the Black-Scholes PDE to identify a set of so-called equivalent economies and then makes a convenient choice of a particular equivalent economy in which valuation is tractable. In the Black-Scholes model, an economy is described by a collection $\{\mathcal{U}, \mu, v, r, s_0\}$ consisting of investor preferences $\mathcal{U}$ (i.e., investor utility functions), an Ito process (3.7) for the objective underlying commodity price dynamics, a risk-free interest rate $r$, and an initial commodity spot price $s_0$ at a beginning date $t_0$. We know that option prices on date $t_0$ depend on $v(\cdot, \cdot)$, $r$, and $s_0$ since they appear explicitly in the Black-Scholes PDE – together with the strike price, expiration date, and any other contractual terms in the associated boundary condition $c_T = c^*_T$ – but option prices do not depend directly on either the underlying commodity’s drift $\mu(\cdot, \cdot)$ or on investor risk preferences $\mathcal{U}$. Because the Black-Scholes model values options based on a risk-free arbitrage argument, neither $\mu$ nor $\mathcal{U}$ affect the arbitrage-free option pricing functions $c$ implicitly described by the Black-Scholes PDE.

This last observation leads to a key insight: Option prices in the true economy $\{\mathcal{U}, \mu, v, r, s_0\}$ are unchanged in any alternate (i.e., imaginary)
Commodity Option Valuation

We consider an economy \( \{U^{alt}, \mu^{alt}, v, r, s_0\} \) with different investor preferences \( U^{alt} \) and spot price drift \( \mu^{alt}(\cdot, \cdot) \) but where the local volatility function \( v(\cdot, \cdot) \), interest rate \( r \), and initial spot price \( s_0 \) are unchanged. This follows since both economies have the same Black-Scholes PDE. Accordingly, we can define a set of equivalent economies with different preferences \( U^{alt} \) and drifts \( \mu^{alt}(\cdot, \cdot) \) in which all option prices are the same as in the true economy.

Options can be priced numerically either by arbitrage (i.e., evaluating the replicating portfolio by solving the PDE) or by Discounted Cash Flows (DCF) valuation. In a particular equivalent economy, the DCF valuation is

\[
c_{jt} = \frac{\mathbb{E}^{alt}[c^*_{jt}]}{1 + \text{RAD}^{alt}_{jt}}
\]

where \( \mathbb{E}^{alt}[c^*_{jt}] \) is the expected option payoff in a particular equivalent economy given the alternative Ito process with drift \( \mu^{alt} \) and \( \text{RAD}^{alt}_{jt} \) is the corresponding alternate risk-adjusted discount rate. We then search for a convenient equivalent economy in which the risk-adjusted discount rate \( \text{RAD}^{alt}_{jt} \) for DCF valuation in that economy (unlike in the true economy) is known a priori. That is to say, we pick alternate preferences \( U^{alt} \) for which we know how to discount in the chosen equivalent economy. Since the Black-Scholes PDE still holds in this chosen equivalent economy, option prices computed there by DCF valuation are, by construction, identical to option prices in the true economy.

A convenient choice of an equivalent economy is a risk neutral economy \( \{U^{RN}, \mu^{RN}, v, r, s_0\} \). With risk neutrality, all expected future cash flows (including those of risky options) are just discounted for time-value-of-money at the risk-free rate. Moreover, the risk-free rate in the equivalent RN economy is the same as in the actual economy by the definition of equivalent economies. For the equivalent RN spot commodity market to clear (i.e., for supply to equal demand), the RN expected return on investible commodities must equal the risk-free rate, \( \mu^{RN} = r \).

\footnote{Discounted Cash Flows values future cash flows as their expected value under the objective measure discounted at the appropriate risk-adjusted discount rate. In contrast, RN valuation values future cash flows as their expected value under the RN measure discounted at the risk-free rate. Given the correct RAD, an option’s DCF valuation will agree with its RN valuation. For more on DCF, see any introductory finance textbook.}
If not, the commodity and bond markets would not clear at the spot price $s_0$ and interest rate $r$ because there would be infinite demand either to buy the commodity and borrow (if $\mu^{RN} > r$) or to short the commodity and lend (if $\mu^{RN} < r$).

To summarize, DCF valuation in the equivalent RN economy implies that the price of any commodity derivative with a payoff $c^*_T$ that is measurable with respect to date $T$ commodity price information is\(^{10}\)

$$c_t = c^{RN}_t = E^{RN}_t[c^*_T]e^{-r(T-t)} \quad (3.13)$$

where the underlying spot commodity price follows the equivalent RN process

$$ds_t = rs_t \, dt + v(s_t, t) \, s_t \, dZ_t. \quad (3.14)$$

We call this the generalized Black-Scholes model for option pricing since it extends the original Black-Scholes model to heteroskedastic RN dynamics. The first equality in (3.13) says that option prices in the true economy are the same as option prices in the equivalent RN economy (by definition of an equivalent economy), and the second equality is DCF valuation in the equivalent RN economy.

Later in this chapter we shall see that a so-called price-of-risk (POR) assumption is needed to identify the RN spot price dynamics in dynamically incomplete markets, but no such POR assumption is needed here in the Black-Scholes RN identification for a dynamically complete market. That is part of the appeal of the Black-Scholes approach. However, many commodities – electricity or natural gas during peak demand times – are not investible assets and, thus, the riskless hedge step in the Black-Scholes identification of the RN dynamics is not possible. As will be seen in Chapter 4, the non-asset nature of such commodities makes the properties of the RN Black-Scholes spot price process (3.14) inappropriate for valuing options on non-asset commodities.

**Commodity futures prices as the underlying variable:** A futures contract with delivery at date $\tau$ specifies a futures price which the long

\(^{10}\)See Shreve [193] for a formal proof of RN valuation that addresses the technicality mentioned in footnote 8.
investor in the futures contract pays to the short investor to buy a unit of a commodity. No money changes hands at initiation, so the futures price \( F_{t,\tau} \) at each date \( t \) is set to make the date \( t \) value of a futures contract with delivery at \( \tau \) equal to zero. In particular, the futures price is not the value of the futures contract; rather it is a contractual term which the market determines so as to set the value of the futures contract to zero.

Over the life of a futures contract, the futures prices on old futures contracts with delivery at \( \tau \) are reset to the new futures prices with compensatory payments between the long and short sides equal, in continuous time, to the instantaneous change in the market futures price \( dF_{t,\tau} \).\(^{11}\) Black [19] developed a riskless hedge argument for pricing options whose terminal payoff \( c^*_T \) at an expiration date \( T \) is measurable with respect to information generated by the futures price \( F_{t,\tau} \) for a specific delivery date \( \tau \geq T \). In other words, the payoff is measurable with respect to the filtration induced by the futures price \( F_{t,\tau} \) alone. In general, this single-futures price filtration is less informative than the full commodity spot price filtration \( F_t \).

The Black argument does not assume that the underlying commodity is an investible asset – i.e., the commodity can be non-storable/perishable such as electricity – but it does assume that the futures contract with date \( \tau \) delivery is traded and that the objective dynamics of the futures price for date \( \tau \) delivery are a general Ito process

\[
dF_{t,\tau} = \mu(F_{t,\tau}, t)F_{t,\tau} \, dt + v(F_{t,\tau}, t)F_{t,\tau} \, dZ_t \tag{3.15}
\]

These dynamics generalize Black [19] which originally assumed the futures price follows a Geometric Brownian Motion. In the Black model, the futures price \( F_{t,\tau} \) is the only underlying random state variable, along with time \( t \), needed to describe its own dynamics. In this sense, the information content of a “state” in the Black model is weaker than in Black-Scholes where the filtration fully describes commodity price dynamics. The Black price at date \( t \) for an option whose payoff \( c^*_T \) is only contingent on futures prices associated with delivery date \( \tau \) is a function \( c(F_{t,\tau}, t) \) of the current futures price \( F_{t,\tau} \) and time \( t \) with ev-

\(^{11}\) In practice, this settlement/payment process happens daily rather than continuously.
3.3. Dynamically Complete Markets

everything else again parametrically, contractually, or historically fixed. To find valuation functions $c$ which are consistent with the absence of arbitrage, we consider a portfolio consisting of a long position in one option combined with a position $-\delta$ in the date-$\tau$ delivery futures contract. Since futures prices are set on date $t$ to make futures contracts have zero market value, this portfolio at date $t$ is worth $c_t - \delta 0 = c_t$. Ito’s Lemma and a variation on the Black-Scholes riskless hedge logic imply that, to avoid arbitrage, the Black partial differential equation must hold

$$\frac{\partial c}{\partial t} + \frac{1}{2} \sigma^2 (F_{t,\tau}, t) F_{t,\tau}^2 \frac{\partial^2 c}{\partial F^2} = c_t r.$$  \hspace{1cm} (3.16)

In the Black PDE, there is no analogue to the $(\partial c/\partial s) s_t$ term in the Black-Scholes PDE since futures contracts, by definition, have a market value of 0 at each date $t$.

Continuing with the RN identification argument, the Black PDE identifies a set of equivalent economies $\{U^{alt}, \mu^{alt}, v, r, F_{t_0,\tau}\}$ with potentially different investor preferences $U^{alt}$ and futures price drift $\mu^{alt}(F_{t,\tau}, t)$ than in the actual economy. Once again, a convenient choice of an equivalent economy is a risk neutral economy $\{U^{RN}, \mu^{RN}, v, r, F_{t_0,\tau}\}$ so that expected option cash flows are just discounted for time-value-of-money at the risk-free rate. Since futures have a zero up-front investment cost, the RN expected futures price drift must be $\mu^{RN} = 0$ for markets to clear at the futures price $F_{t_0,\tau}$. Otherwise, risk-neutral investors would try to take unboundedly large long or short futures positions and markets in the equivalent RN economy would not clear at the current futures price $F_{t,\tau}$ (which would contradict part of the definition of an equivalent RN economy).

To summarize the key points about the generalized Black model, commodity derivatives are priced as discounted RN expectations

$$c_t = \mathbb{E}^{RN}_t [c_T] e^{-r(T-t)}$$  \hspace{1cm} (3.17)

where the underlying RN futures price dynamics are

$$dF_{t,\tau} = v(F_{t,\tau}, t) F_{t,\tau} dZ_t.$$  \hspace{1cm} (3.18)

\footnote{The replicating equation with futures which is analogous to (3.11) is $dc_t = (\partial c/\partial F) dF_{t,\tau} + c_t r dt$. This says that the change in the option price here is replicated by a position $\partial c/\partial F$ in the futures contract and $c_t$ dollars in t-bills.}
Although similar to Black-Scholes, a significant limitation of the Black model is that it can only be used to price options with payoffs \( c_T \) which are functions measurable with respect to the sub-filtration induced by a particular futures price \( F_{t,\tau} \) with a particular delivery date \( \tau \) whose dynamics are given in (3.18). This is in contrast to Black-Scholes which can price any option payoff \( c_T^* \) that is measurable with respect to the full commodity price process filtration. The Black model for a given futures price \( F_{t,\tau} \) cannot be used – without additional assumptions – to price options on the spot commodity at expiration dates other than date \( \tau \) (at which time the instantaneous futures price \( F_{\tau,\tau} \) equals the spot price \( s_\tau \)). More generally, the Black model cannot price options whose payoffs depend on information defined by the filtration for spot prices generally (e.g., paths of spot prices) or on futures prices \( F_{t,\tau'} \) for other delivery dates \( \tau' \neq \tau \). To extend the Black approach to price general payoffs measurable with respect to commodity spot prices at any date, or futures prices for arbitrary delivery dates, requires a model of the dynamics of the entire term structure of futures prices (see Section 4.3 in Chapter 4).

Using a commodity option to identify RN dynamics: The reason for the limitations on option pricing in the single-futures contract Black approach is that the connection from the single futures price (whose RN dynamics \( dF_{t,T} \) are known a priori) back to the more primitive state space induced by the underlying RN commodity spot price dynamics is not known. In other words, the futures price function \( F_{t,\tau}(s_t, t) \) relating futures prices \( F_{t,\tau} \) and commodity spot prices \( s_t \) is not known. However, if this relation is monotone and known a priori, then we can again uniquely identify the RN commodity price dynamics. Moreover, this point is not limited just to futures. It applies to all derivatives subject to a natural monotonicity condition. To see this, suppose that the objective dynamics of the underlying commodity spot price – where the commodity is not an investible asset – are known to be

\[
d s_t = m(s_t, t)dt + v(s_t, t)dZ_t
\]

and that there is an option with expiry \( T \) with a known pricing function \( c(s_t, t) \) relating the option valuation and the spot price where \( \partial c/\partial s \neq 0 \).
From Ito’s lemma, the dynamics of the known option price are
\[
d c_t = \left[ \frac{\partial c}{\partial t} + \frac{1}{2} v^2(s_t, t) \frac{\partial^2 c}{\partial s^2} + \frac{\partial c}{\partial s} m(s_t, t) \right] dt + \frac{\partial c}{\partial s} v(s_t, t) dZ_t. \tag{3.20}
\]

Given these assumptions, the question is: What are the RN dynamics
\[
d s_t = m^{RN}(s_t, t) dt + v^{RN}(s_t, t) dZ_t \tag{3.21}
\]
for commodity spot price? Since the quadratic variation of an Ito process is unchanged under the RN measure, we have \(v^{RN}(s_t, t) = v(s_t, t)\).\(^{13}\) Turning to the RN drift, since options are assets, the known option’s RN expected return is the risk-free rate; which implies
\[
\frac{\partial c}{\partial t} + \frac{1}{2} v^2(s_t, t) \frac{\partial^2 c}{\partial s^2} + \frac{\partial c}{\partial s} m^{RN}(s_t, t) = r c(s_t, t). \tag{3.22}
\]
Rearranging this gives
\[
m^{RN}(s_t, t) = \frac{r c(s_t, t) - \partial c/\partial t - (1/2) v^2(s_t, t) \partial^2 c/\partial s^2}{\partial c/\partial s} \tag{3.23}
\]
where, since the function \(c(s_t, t)\) is known, all of the partial derivatives on the righthand side of (3.23) are known, and, thus, the RN commodity price drift is determined. Given the RN spot price dynamics, we can now use numerical RN valuation, as in equation (3.5), to value any stream of commodity-linked cash flows over time that are measurable with respect to the commodity price process. (In addition, the known option can be used to hedge spot price induced randomness \(d s_t\) in any other commodity-linked derivative.)

Unfortunately, it is rare that the option valuation function \(c\) is known a priori for even one option. As a result, this approach and the RN drift identification in (3.23) is typically not of practical use. However, this discussion does illustrate some of the subtleties in the connection between spot prices and derivative prices when identifying RN spot price dynamics.

\(^{13}\)See Shreve [193].
3.4 Dynamically Incomplete Markets

Markets are dynamically incomplete when the underlying variable for an option is driven by Brownian Motions, jumps, or auxiliary state variables which are not tradable. For example, suppose that the spot prices for a commodity follow a jump-diffusion process

\[ ds_t = m(x_{t-}, s_{t-}, t)dt + v(x_{t-}, s_{t-}, t)dZ_t + \xi(x_{t-}, s_{t-}, t)dJ_t \]  

(3.24)
driven by a (continuous) Standard Brownian Motion \(dZ_t\) and (discontinuous) Poisson or Compound Poisson jumps \(dJ_t\). The spot price local drift \(m(x_{t-}, s_{t-}, t)\), diffusion local volatility \(v(x_{t-}, s_{t-}, t)\), and the predictable jump magnitudes \(\xi(x_{t-}, s_{t-}, t)\) and jump probability intensities \(\lambda(x_{t-}, s_{t-}, t)\) can depend on a vector of economic and environmental state variables \(x_t\) (with their own Ito or jump-diffusion dynamics \(dx_t\)) as well as on the spot price \(s_{t-}\) and time \(t\). In such a setting, option prices are functions of \(s_{t-}\), \(x_{t-}\), and \(t\), and also, potentially, of other market preference factors affecting risk premia (e.g., market wealth portfolio returns).

The problem with non-traded randomness in spot price dynamics is that there are no market prices from which to infer the market preferences which adjust the objective probabilities into RN probabilities. Equivalently, we cannot construct riskless hedged option/underlying portfolios as in the Black-Scholes RN identification. In a dynamically

---

14 The discussion here is exposited in terms of the physical availability of securities with which to trade the state variables and jumps. A market can also be epistemologically incomplete if sufficient securities are physically traded, but their dynamics are not known to the individual wanting an option pricing model. In dynamically complete markets, simply knowing the current prices of traded securities is not sufficient for a market to be dynamically complete. The future dynamics of the traded securities must also be known in order to recover the implicit underlying RN dynamics. This generalizes the point made regarding the RN identification in (3.23) in the previous subsection.

15 Jump processes lead to discontinuities in the spot price process and possibly in the state variables. The \(t^\) notation means that local drifts, volatilities and the jump moments do not depend on the spot price \(s_t\) and the state variables \(x_t\) at exactly date \(t\) (which would include any unpredictable jumps in the state variables occurring at date \(t\)) but rather on the limiting values \(\lim_{\tau \to t^-} s_{\tau}\) and \(\lim_{\tau \to t^-} x_{\tau}\) of the spot price and state variables as they approach date \(t\). The realized magnitudes of the jumps \(dJ_t\) can be either known in advance (e.g., normalized to 1 so that \(\xi(x_{t-}, s_{t-}, t)\) is the jump size) or they can be i.i.d. random variables (normalized with a mean 1 so that \(\xi(x_{t-}, s_{t-}, t)\) is the conditional expected jump size). See Chapter 4 and Shreve [193] for more on Poisson and Compound Poisson processes.
incomplete market, option pricing models therefore impose specific structure on market risk preferences in order to identify (compute) the RN dynamics. These so-called *price-of-risk (POR)* assumptions specify the difference between objective and RN probabilities for the underlying commodity price dynamics. The same POR are then assumed to be consistently embedded in all derivative prices. One model of incomplete market option pricing differs from another model because of the different particular POR assumptions they make.

Suppose again that the underlying commodity is a non-asset and that, while there may be observed market *prices* for some commodity derivatives, there is no traded derivative whose price *function* is known *a priori*. In this case, there is not enough information to pin down the RN spot price dynamics, and derivatives can only be priced given an auxiliary POR assumption. For example, if objective spot price dynamics are given by (3.19), then a common pricing procedure is to assume RN dynamics

\[ ds_t = m^{RN}(s_t, t) \, dt + v(s_t, t) \, dZ_t, \]

where the RN drift is

\[ m^{RN}(s_t, t; \theta) = m(s_t, t) + \pi(s_t, t; \theta) \]

where the price-of-risk \( \pi(s_t, t; \theta) \) is typically assumed to have a functional form that makes the RN dynamics qualitatively similar to the objective dynamics (which can be empirically estimated) and where \( \theta \) is a set of parameters which must be calibrated (see §3.5). In contrast to RN asset drifts, non-asset RN drifts have no necessary relation to the risk-free interest rate \( r \).

Although markets cannot be dynamically complete with just a non-investible commodity and t-bills, once we can condition on an assumed POR, the market may be complete given tradable derivatives. Given a particular POR in (3.26), and the assumption of no jumps in (3.25), any option with a non-zero sensitivity \( \partial c/\partial s \) can be used to dynamically synthesize and price any other measurable commodity state-contingent cash flow.\(^{16}\) Of course, this only works if the POR assumption is correct.

\(^{16}\)This result can be further generalized to allowing for RN spot price dynamics that depend on multiple factors \( x_t \), driven by additional Brownian Motions and a set of options with non-zero linearly independent sensitivities \( \partial c/\partial s \) and \( \partial c/\partial x_t \).
Thus, dynamic replication based on an assumed ad hoc POR is subject to potential model error risk.

### 3.5 Calibration

Once the functional form of the RN dynamics has been specified, it still must be parametrically calibrated. There are two ways RN dynamics are calibrated. If parameters are the same under both the objective and RN measures, then one approach is to estimate them statistically using historical data. This, of course, assumes that the objective data generating process in the future is the same as it was in the past. Statistically estimated parameters are also estimated with error. A widely-used alternative approach is *implied calibration*. If $\theta$ denotes the parameters of the RN dynamics,

$$
\begin{align*}
    ds_t = m^{RN}(s_t, t; \theta) \, dt + \sigma^{RN}(s_t, t; \theta) \, dZ_t,
\end{align*}
$$

(3.27)

then the prices of traded commodity-contingent securities depend implicitly on $\theta$

$$
\begin{align*}
    c_t(\theta) = E^{RN}_{t} [c^*_{T}; \theta] e^{-r(T-t)}.
\end{align*}
$$

(3.28)

Implied calibration involves solving numerically for values of $\theta$ that equate one or more model prices $c_t(\theta)$ with observed market security prices $c^M_{t,T}$.

### 3.6 Summary

This chapter has reviewed how commodity RN price dynamics are identified and the connection with market completeness. A key idea implicit in this discussion is what defines a “state” for commodity price dynamics. The generalized Black-Scholes model assumes that the spot price and date $(s_t, t)$ constitute the only state variables necessary for describing RN commodity spot price dynamics. Given this assumption, any $\mathcal{F}_T$-measurable commodity option payoff at any expiration date $T$ can be priced. In contrast, a coarser set of states in the single-futures price Black model are defined based on time and the futures price for a particular fixed delivery date $\tau$. Given this coarser definition of the driving state variables, $(F_{t, \tau}, t)$, the Black model can value option payoffs that
are measurable with respect to a restricted filtration induced just by the particular futures price $F_{t, \tau}$ for a particular delivery date $\tau$.

When the spot price (or futures price) dynamics depend on non-asset factors other than just the spot price (or futures prices) themselves, then price-of-risk assumptions are usually needed for the identification of any RN non-asset factor processes. This issue is discussed further in the context of specific multi-factor models in Chapter 4.

3.7 Notes

The existence, uniqueness, and identification of state prices, or equivalently RN probabilities, is more general than the application to commodities. In particular, the same identification issues arise when pricing options in incomplete markets for other non-asset underlying variables. This includes option valuation when the underlying variables are interest rates, weather, volatility, and events such as corporate defaults.

The general theory for dynamic state-contingent claim valuation was first worked out in Harrison and Kreps [113]. Duffie [81] is a more recent exposition. Continuous-time models of option pricing rely on the mathematics of stochastic calculus. Shreve [193] provides a systematic introduction to Brownian motions, Ito processes, jump processes, and their application to option pricing. When option valuations do not have analytic expressions, option price are computed numerically either using tree and lattice methods or via Monte Carlo simulation. Hull [119] gives a description of tree and finite difference methods. Glasserman [103] is a comprehensive reference on Monte Carlo methods in finance.

Our discussion of option pricing in incomplete markets implicitly assumes that introduction of a new option (which needs to be priced) does not change the fundamental economics of valuation. This is a non-trivial assumption. In general, introducing new (non-redundant) securities in an incomplete market increases the span of tradable states which, in turn, can potentially change investor asset demands and, thus, can change market-clearing traded securities prices. In this case, the new option needs to be priced to be consistent with the new post-option-introduction traded security prices, not the pre-introduction prices.

A related set of issues concerns option pricing when liquidity is
priced. In this case, the state prices implicit in actively traded securities may include a premium for liquidity which would not carry over to the valuation of illiquid real options. Vayanos and Wang [214] is a general survey on liquidity and asset pricing. Lastly, it should be noted that Black-Scholes and Black both assume trade occurs in frictionless markets (i.e., no bid-ask spreads and transaction costs). Option pricing with frictions is generally difficult. Kabanov and Safarian [123] discuss option pricing (and consumption-investment) models with transaction costs in trading.
Modeling Commodity and Energy Prices

4

This chapter reviews specific models used for commodity derivative valuation. Section 4.1 introduces the microeconomics of market-clearing commodity prices and their dynamics. Sections 4.2 and 4.3 review reduced-form spot price evolution models and reduced-form modeling of the term structure of futures prices. In practice, these are the most common approaches used to model commodity prices. Section 4.4 gives an overview of a third modeling approach, equilibrium pricing, which is less tractable but more conceptually grounded in the underlying economics of commodities. Section 4.5 gives a brief review of the empirical evidence on commodity prices. Section 4.6 summarizes. Section 4.7 gives general pointers to the literature.

4.1 Introduction

The valuation of options and derivatives of any type requires specifications for the dynamics of the underlying state variables driving the payoff cash flows. This is true both for financial options and commodity conversion assets. In the case of real options to physically store and transform commodities, the spot and futures prices at which commodi-
Commodity and Energy Prices

D(s; xD\_t, t)

s_t

Q_t

Quantity

Price

S(s; xS\_t, t)

D(s; xD\_t, t)

Fig. 4.1 Commodity pricing and supply and demand.

ties can be bought and sold are natural state variables. Indeed, some option pricing models (e.g., Black-Scholes in Chapter 3) take the spot price as the only state variable. More fundamentally, however, any factor which affects the dynamics of future commodity supply and demand curves, and, thus of future spot prices, is a state variable. Thus, much of this chapter will focus on multi-factor models of option pricing.

Commodity price processes are defined formally on a probability space \((\Omega, \mathcal{F}, P)\) where \(\Omega\) is the set of all possible states, \(\mathcal{F} = \{\mathcal{F}_t\}\) is a filtration which describes the possible evolution over time of information about the realized state \(\omega \in \Omega\), and \(P\) is a probability measure. More concretely, the dynamics of commodity prices are induced by the dynamics of supply and demand. In contrast to the more abstract treatment in Chapter 3, this chapter interprets specific models in terms of their ability to represent reasonable supply- and demand-driven commodity price dynamics.

From standard microeconomics, the spot price \(s_t\) of a commodity at date \(t\) is the market-clearing price which equates physical supply and demand. This is illustrated in the familiar diagram in Figure 4.1. The demand curve \(D(s; xD\_t, t)\) is the aggregate quantity of the physical commodity demanded at each possible price \(s\) on date \(t\) given the set of factors \(x\_D\_t\) which affect demand. Demand is the sum of demands for immediate physical use plus, if the commodity is storable, any inven-
4.1. Introduction

45

The supply schedule $S(s; x^S_t, t)$ is the aggregate quantity of the physical commodity supplied at each price $s$ on date $t$ given the set of factors $x^S_t$ which affect supply. Supply is the sum of current production plus any commodity stocks available from inventories stored in the past. The levels, slopes, and functional forms of the supply and demand curves can change over time depending on factors such as weather, macroeconomic conditions, and the prices of other commodities which are complements or substitutes in consumption and/or in production. In equilibrium, aggregate supply equals aggregate demand at the equilibrium quantity,

$$D(s_t; x^D_t, t) = S(s_t; x^S_t, t) = Q(x_t, t),$$

at the market-clearing price

$$s_t = s(x_t, t),$$

where $x_t = (x^D_t, x^S_t)$ is the combined set of demand and supply factors. The dynamics of market-clearing spot prices $ds_t$ are induced by the shapes of the supply and demand schedules, as reflected in the spot price function $s(\cdot, \cdot)$, and by the dynamics of the supply and demand state variables $dx_t$.

The stochastic processes followed by the state variables $x_t = (x^D_t, x^S_t)$ are typically represented as a system of stochastic differential equations like the following:

$$dx_{k,t} = [m_k(x_{t-}, t) - \xi_k(x_{t-}, t)\lambda(x_{t-}, t)]dt + v_k(x_{t-}, t) dZ_t + \xi_k(x_{t-}, t) dJ_t, \quad \forall k = 1, \ldots, K,$$

where the instantaneous change $dx_{k,t}$ in the $k$-th state variable at date $t$ is driven by potentially three building blocks: The deterministic passage of time $dt$, the random instantaneous increments of a vector of Standard Brownian Motions $dZ_t$, and the random instantaneous increments of a vector of Poisson or Compound Poisson jump processes $dJ_t$ with jump probability intensities $\lambda(x_{t-}, t)$. The corresponding jump magnitudes are either fixed (and normalized to 1 so that $\xi_k(x_{t-}, t)$ is the vector of local jump sizes for the $k$-th state variable) or are independently and identically distributed random vectors $Y_t$ (with mean vector $E_t[Y_t]$ normalized to the unit vector so that $\xi_k(x_{t-}, t)$ gives the expected local jump sizes for the $k$-th state variable) defined at a
discrete set of jump dates (i.e., dates on which $dJ_t \neq 0$). The passage of time $dt$ affects the state through changing seasons and other predictable changes (e.g., required lead times in bringing capacity on line or predictable components of macroeconomic or demographic variables). Since Brownian Motions are continuous, the increments $dZ_t$ represents the gradual arrival of incremental news. The jumps $dJ_t$ represent the arrival of dramatic news which causes discontinuous changes in the state variables and, thus, in prices.

The magnitudes of the impacts of the three building blocks $(dt, dZ_t, dJ_t)$ on each of the state variables $dx_{k,t}$ are determined by the expressions multiplying them: $m_k(x_t, t)$ is the predictable local drift, $v_k(x_t, t)$ is a vector of loadings which scale the local volatilities of the Standard Brownian Motions, and $\xi_k(x_t, t)$ is a vector which scales the jump magnitudes. The local drifts, volatility, and jump intensities and magnitudes can all change over time depending on the state variables $x_t$ and on time $t$. Time-dependence in price levels, volatilities, and other dynamics can be due to predictable seasonalities (e.g., harvests or weather patterns) and predictable macroeconomic, demographic, or technological trends (e.g., time to build new power plants or technology adoption dynamics). The $t^-$ notation means that drifts, volatilities and the jump moments do not depend on state variables at exactly date $t$ (which would include any unpredictable jumps in the state variables occurring at date $t$) but rather on the limiting value of the state variables as they approach date $t$. The term $\xi_k(x_t, t)\lambda(x_t, t)$ in the drift compensates for the contributions of the expected Poisson jumps so that $m_k(t, x_t^-)$ is the total expected change.

The state variable dynamics $dx_t$ and the equilibrium spot price function $s(\cdot, \cdot)$ jointly induce the dynamics of the spot price $ds_t$. Figures 4.2 through 4.4 show historical daily spot prices for gold, natural gas, and power. The clear differences between the plots illustrate that the supply and demand dynamics for different commodities can differ substantially. Gold prices look like the prices of investible financial assets, 

---

1 See Shreve [193] for properties of Brownian motions and Poisson processes.
2 The total volatility of the state variables reflects both the scaled Brownian Motion randomness and the scaled jump randomness.
such as equity, in that they appear to have persistent shocks that cause them to wander up or down. In contrast, natural gas prices appear to be mean-reverting, and power prices appear to be even more strongly mean-reverting. Other common empirical properties of spot prices for various commodities include seasonality in price levels, seasonality and other forms of heteroskedasticity in volatility, and jumps.

The statistical properties of commodity spot price processes have a microeconomic foundation in that they are induced by the statistical properties of the underlying supply and demand dynamics. This is illustrated in Figure 4.5. Factors which cause predictable seasonal changes in the levels of physical supply and demand (e.g., seasonal differences in average temperature or harvest cycles) can induce seasonalities in market-clearing spot prices. Factors which cause supply and demand curves to fluctuate around a typical shape induce mean-reversion in spot prices (e.g., mean-reversion in temperature around a seasonal average temperature or random equipment failures which initially reduce supply but then are repaired). Factors which cause supply and demand
Fig. 4.3 Evolution of the Henry Hub natural gas spot price between 2003 and 2012. Source: US Energy Information Administration.

Fig. 4.4 Evolution of the PJM Western Pennsylvania weighted average electricity spot price between 2003 and 2012. Source: US Energy Information Administration.
4.1. Introduction

Fig. 4.5 Supply and demand and the probability density functions of commodity prices and equilibrium quantities.

to become more or less elastic/inelastic (as shown in Figure 4.5) induce heteroskedasticity in spot prices. Factors which cause abrupt jumps in supply and demand (e.g., infrastructure failures) induce jumps in spot prices.

Different option pricing models make different assumptions about the number and type of state variables that drive the objective commodity price filtration and the associated preference-adjusted RN dynamics. These different model assumptions lead to different option valuations. In particular, some models take the spot price and time, \((s_t, t)\), as the sole state variables. Other models represent spot prices as a combination of multiple abstract statistical factors driving both price levels and price volatility. Still other models use explicitly identified factors – e.g., weather, equipment capacity, or macroeconomically-driven demand – as causal state variables.

An important property of a commodity is whether the good is priced as an investible asset or as a non-asset. As discussed in Chapter 3, assets have RN drifts equal to the risk-free interest rates, but non-asset RN drifts have no necessary connection to the risk-free rate. The distinction
between asset and non-asset commodities will be central in Section 4.2.

**Forward and futures prices:** Forward contracts are financial derivatives which, as discussed in §2.2 in Chapter 2, allow the long side of a forward trade to buy a commodity from the short side of the trade at a forward price $f_{t,T}$ to be paid on a future delivery date $T$. In particular, the market forward price $f_{t,T}$ on date $t$ is contractually set so that both the long and short sides are willing to enter into the forward contract with no money changing hands at date $t$. There are no subsequent cash flows until delivery when the contract is settled with an exchange of value (from the long’s perspective) of $s_T - f_{t,T}$. This exchange of value can be through physical delivery of the commodity itself or via a cash settlement through a net cash payment.

Futures contracts are similar to forward contracts except that futures settle each day $t_i \leq T$ with a cash flow between the long and short sides equal to the difference $F_{t_i,T} - F_{t_{i-1},T}$ between the prevailing futures price on date $t_i$ for delivery at $T$ and the futures price from the day before. Thus, the only contractual difference between futures and forwards is the timing of settlement. A useful theoretical result in this context is that if the risk-free interest rate is non-random – an assumption which is often made for commodity option pricing given that commodity price volatility is empirically much larger than interest rate volatility – then arbitrage-free futures and forward prices are identical.\(^3\)

On any given date $t$, the **futures curve (or futures term structure)** is the collection $F_t = \{F_{t,T}\}$ of futures prices for all traded future delivery dates $T$. If the futures curve is rising with time-to-delivery, then the futures term structure is said to be in **contango**. If the curve is falling in time-to-delivery, then the curve is said to be **backwardated**. In natural gas markets, it is common for futures curves to be both increasing and decreasing between various different delivery dates because of predictable seasonalties in weather-driven peak demands for natural gas in winters (for heating) and sometimes in summers (for air conditioning).

\(^3\)See Hull [119] for more on futures and forwards.
4.1. Introduction

Figure 4.6 shows the evolution of daily natural gas futures curves over the decade of 2003-2012. As time $t$ passes and information flows into the market, the futures price $F_{t,T}$ for a given delivery date $T$ changes to reflect the impact of arriving persistent information. The seasonal humps in the natural gas futures curves create oncoming “waves” as time $t$ passes and the time-to-delivery $T - t$ gets shorter for fixed delivery dates $T$. The impact of increased demand for natural gas over the decade and the impact of the shale gas boom at the end of the decade are readily apparent in the changing average price levels over time.

Futures prices can differ from expectations $E_T[s_T]$ of future spot prices, under the objective measure, because of risk premia. For example, Keynes [134] describes how imbalances between heterogeneous speculators and hedgers can lead to premia and discounts in futures prices relative to objective future spot price expectations. In this case, the expected change in the futures price for a fixed delivery date $T$ can have a non-zero statistical drift. In contrast, under the RN measure,
futures prices are RN expectations conditional on date $t$ information of future date $T$ spot prices:

$$F_{t,T} = \mathbb{E}_t^R[s_T]$$

(4.3)

and, thus, futures prices are martingales under the RN measure:

$$\mathbb{E}_t^R[dF_{t,T}] = 0.$$  

(4.4)

The martingale property of RN futures price dynamics follows from iterated expectations and the daily settlement mechanics.

### 4.2 Spot-price Evolution Models

One widely-used type of reduced-form model is the spot price evolution approach where the RN spot price dynamics are modeled without explicit reference to the underlying microeconomic drivers.

**Black-Scholes and further generalizations:** In §3.4, we saw that the local Black-Scholes drift can be any function $\mu(s_t, t)$ under the objective measure, but that the RN drift is the risk-free rate $r$ (see equations (3.7) and (3.14)). While possibly a reasonable approximation for stock prices and asset-like commodities (e.g., gold), these dynamics are problematic for non-asset commodities. In particular, the generalized Black-Scholes RN dynamics have constant expected returns equal to the risk-free interest rate, no seasonality, and no jumps. In contrast, actual commodity spot prices often exhibit mean-reversion, seasonalties, and jumps. While these statistical properties are characteristics of the objective dynamics observed in real-world price data, they also seem to carry over to the RN dynamics since it is often difficult, in practice, to calibrate the generalized Black-Scholes model to reproduce seasonality in market futures curves and/or some term structures of volatility in commodity option prices.

The Black-Scholes model has strong implications for the futures curve. Since the RN drift of spot prices is the risk-free rate, futures curves implied by Black-Scholes are always in contango and rise with time-to-delivery at the risk-free rate:

$$F_{t,T} = s_t e^{r(T-t)}.$$  

(4.5)
However, as can be seen in Figure 4.7, commodity futures prices in the market can increase and decrease with time-to-delivery in ways that are at odds with the Black-Scholes contango curves. The Black-Scholes model also restricts random shocks to the futures curves, under the RN measure, to proportional parallel shifts:

$$\frac{dF_{t,T}}{F_{t,T}} = v(s_t, t) dZ_t,$$

(4.6)

since neither the local spot volatility $v(s_t, t)$ from (3.7) in Chapter 3 (which gives the local futures volatility for each maturity $T$) nor the standard Brownian Motion shock $dZ_t$ depend on the futures delivery date $T$ and, thus, are the same for each futures price.

Subsequent research in commodity option pricing has worked to develop tractable models with more realistic commodity price dynamics. One variant of the Black-Scholes RN dynamics includes a deterministic local convenience yield, denoted by $y(t)$, in the spot price drift:

$$ds_t = [r - y(t)] s_t \, dt + v(t, s_t) s_t \, dZ_t.$$

(4.7)
The convenience yield $y(t)$ is like a non-monetary dividend representing a flow of services assumed to accrue to the owners of physical commodities. This variant of Black-Scholes can then be calibrated to fit any initial commodity futures curve by recursively choosing the convenience yields to set RN expected future spot prices at each delivery date equal to the corresponding market futures price:

$$F_{t,T} = s_t e^{r(T-t) - \int_t^T y(\tau) d\tau}.$$  \hfill (4.8)

If the convenience yield is greater than the risk free rate, $y(T) > r$, then the futures curve will be locally decreasing in maturity at delivery date $T$ (i.e., the slope will be negative) and if $y(T) < r$, then the futures curve will be locally increasing in maturity at delivery date $T$. However, conditional on the level of the realized spot price $s_t$, the associated futures curve on date $t$ is not random. In addition, random shocks to the futures curve again lead to parallel shifts.

The Black-Scholes model also has strong implications for the term structure of volatility. The volatility of the cumulative log spot price change over a time interval of length $T - t$ is

$$\text{var}_t \left[ \ln(s_T/s_t) \right] = \text{var}_t \left[ \int_{\tau=t}^T d \ln(s_{\tau}) \right] = \text{var}_t \left[ \int_{\tau=t}^T v(t, s_{\tau}) dZ_t \right] = \int_{\tau=t}^T v_{t,\tau}^2 d\tau,$$

where (i) the first equality follows from the fact that log changes are additive, (ii) the second equality follows from the fact that spot price changes are serially uncorrelated over time in Black-Scholes, (iii) and the third equality defines $v_{t,\tau}^2$ as the variance of the instantaneous date $\tau$ log spot price changes conditional on date $t$ information.\textsuperscript{4} This variance decomposition implies that cumulative price change variances are weakly increasing in $T - t$ in generalized Black-Scholes. In contrast, if the (objective and RN) price changes are negatively autocorrelated (e.g., as with a mean-reverting price process), then heteroskedasticity and/or changing speeds of mean-reversion can cause the cumulative return variance – both empirically and implied from market option prices – to be non-monotone in the time interval length $T - t$.

\textsuperscript{4}The conditional variances, $v_{t,\tau}^2$, are different from the future local variances, $v(\tau, s_{\tau})$, in that the $v(\tau, s_{\tau})s$ can depend on prevailing future prices $s_{\tau}$ whereas the $v_{t,\tau}^2s$ can only depend on the spot price $s_t$ at date $t$ but not on the realized future spot prices.
Another even more general variant of Black-Scholes allows the convenience yield to depend on the prevailing spot price level as well as on time:

$$ds_t = [r - y(t, s_t)] s_t dt + v(t, s_t) s_t dZ_t. \tag{4.10}$$

Now the shape of the futures curve can change stochastically in ways that are empirically more realistic. For example, high spot prices can be associated with local backwardation at the short end of the futures curve if $y(t, s_t) > r$ when $s_t$ is high and with local contango if $y(t, s_t) < r$ when $s_t$ is low. However, this randomness is driven entirely by the spot price, which means, again, that knowing $s_t$ and the date $t$ is sufficient to know the shape of the futures curve.

While convenience yields improve the fit of Black-Scholes to market commodity derivative prices, there are two conceptual problems with this modeling approach. First, when the underlying variable is a traded asset, the Black-Scholes RN dynamics are derived from a no-arbitrage argument without price-of-risk assumptions. However, the no-arbitrage argument is valid only if the underlying variable is a traded asset. Consequently, when the various variants of Black-Scholes are applied to non-asset commodities, the RN dynamics in (3.14), (4.7), and (4.10) are effectively assumed, not derived. In this sense, while the mathematics of the RN dynamics can be assumed to have a generalized Black-Scholes form, the conceptual justification for these RN dynamics is absent when the Black-Scholes model is used to price non-asset commodity options.

The second problem specifically concerns convenience yields. Unlike stocks which pay actual dividends, ownership of a barrel of oil or an mmBtu of natural gas does not pay any interim cash flows to commodity owners. Commodity ownership may entail storage costs (negative dividends), but no positive cash flows accrue to owners of commodities over time. Similarly, a short position in equity involves paying dividends, but short positions in commodities do not receive any negative dividends. In short, convenience yields may be a convenient model-fitting device, but pseudo-dividends have no physical basis in cash flows in the real world to put any restrictions on the commodity RN drift.

The rest of this section presents a variety of RN spot price dynamics used in other option pricing models which have been proposed as
alternatives to Black-Scholes given its potential problems as a model of non-asset commodity price dynamics.

**Gibson-Schwartz:** The Gibson and Schwartz [102] model is a well-known two-factor model of RN spot price dynamics:

\[
\begin{align*}
    ds_t &= [r - y_t] s_t \, dt + \sigma s_t \, dZ_{s,t}, \\
    dy_t &= \alpha[\theta - y_t] \, dt + \xi \, dZ_{y,t}.
\end{align*}
\]  

(4.11)

Spot prices follow a process similar to a Geometric Brownian Motion except that now a stochastic convenience yield is subtracted from the the risk-free rate in the RN spot price drift. The convenience yield follows an Ornstein-Uhlenbeck process which mean-reverts to a long-run mean of \( \theta \) which includes both the objective long-run mean plus a possible risk premium. The Standard Brownian Motions \( dZ_{s,t} \) and \( dZ_{y,t} \) driving the two processes can be correlated as needed. The Brownian Motion \( dZ_{s,t} \) represents persistent shocks to commodity supply and demand, while \( dZ_{y,t} \) changes how much the mean-reverting convenience yield causes spot prices to deviate from constant-drift paths.

In terms of fitting observed market derivative prices, this approach has additional flexibility relative to the deterministic and the Markov convenience yield versions of Black-Scholes in (4.7) and (4.10) in that, given the spot price, the shapes of futures curves are now random and not perfectly correlated with prevailing spot price. The short end of the futures curve will be locally increasing (decreasing) if the current convenience yield \( y_t \) is less (greater) than the risk-free rate \( r \). Whether or not the futures curve is increasing or decreasing at distant future delivery dates depends on whether the long-run mean convenience yield \( \theta \) is less than or greater than \( r \). Once again, as in the generalized Black-Scholes model, since the RN commodity price process appears to have a RN drift equal to the risk-free rate after adjusting for the convenience yield – i.e., the same RN drift as for asset prices – the suggestion that commodity prices are asset prices is a somewhat misleading intuition for non-asset commodities.

**Affine models:** Schwartz [182] extends the two-factor Gibson and Schwartz [102] model to a three-factor model which also includes a
4.2. Spot-price Evolution Models

Spot interest rate process. Casassus and Collin-Dufresne [47] further extend the specification of multi-factor spot rate models and investigate their empirical fit. In particular, they assume that log commodity spot prices under the RN measure follow a maximal affine function of three mean-reverting state variables. With the additional identifying assumptions that (i) the stochastic risk-free spot interest rate is an Ornstein-Uhlenbeck process:

$$dr_t = \alpha_r (\theta_r - r_t) \, dt + \sigma_r \, dZ_{r,t}, \quad (4.12)$$

and (ii) commodities are traded assets with convenience yields, the resulting RN dynamics for spot prices and convenience yields are

$$dy_t = [\kappa_{y,0} + \kappa_{y,r} r_t + \kappa_{y,y} y_t + \kappa_{y,s} \ln(s_t)] \, dt + \sigma_y \, dZ_{y,t}, \quad (4.13)$$

$$ds_t = [r_t - y_t] \, s_t \, dt + \sigma_s \, s_t \, dZ_{s,t},$$

where $\alpha_r, \theta_r, \kappa_s, \sigma_s$ are model parameters. In this model the convenience yield is not just correlated with the spot price via potentially correlated standard Brownian Motions $dZ_{y,t}$ and $dZ_{s,t}$; rather now the convenience yield dynamics depend directly on spot prices as a factor.

**Pilipovic/Schwartz-Smith:** Pilipovic [167] and Schwartz and Smith [183] dispense with treating imperfectly storable commodities as investible assets with RN expected returns equal to the risk-free interest rate. Instead, they simply posit an incomplete market setting in which the RN spot price dynamics include permanent and transitory components:

$$s_t = L_t \, e^{\epsilon_t}, \quad (4.14)$$

$$dL_t = \mu^{RN} L_t \, dt + \sigma L_t \, dZ_{L,t},$$

$$d\epsilon_t = \alpha[\theta - \epsilon_t] \, dt + \xi \, dZ_{\epsilon,t},$$

where the two standard Brownian Motions, $dZ_{L,t}$ and $dZ_{\epsilon,t}$, are potentially correlated. The Geometric Brownian Motion $L_t$ represents persistent long-run price trends due to persistent shocks to long-term supply.

---

5Dai and Singleton [63] define an affine model as being “maximal” if it has the maximum number of identifiable parameters given futures prices alone.

6Ross [176] models RN spot prices with mean-reversion but no persistent shocks.
and demand (e.g., technological innovations or discoveries of new deposits). The mean-reverting process $\epsilon_t$ then causes the market-clearing spot price to wander away from the long-term price due to short-lived supply and demand shocks (e.g., weather). Futures prices are available in closed-form:

$$F_{t,T} = L_t \exp \left( e^{-\alpha(T-t)} \epsilon_t + [1 - e^{-\alpha(T-t)}] \theta + A(T - t) \right),$$

where $A(T - t)$ is a function of the futures time-to-maturity that depends on the other parameters of the model. Note that, although $L_t$ and $\epsilon_t$ are unobserved latent variables, once the model parameters are known, the two factors can be recovered (i.e., implied) from any two futures prices.

One significant fact – proven in Schwartz and Smith [183] – is that the Gibson-Schwartz and Pilipovic/Schwartz-Smith models are mathematically equivalent “rotations” of each other. For each calibration of the Pilipovic/Schwartz-Smith RN dynamics in (4.14), there is a mathematically equivalent calibration of the Gibson-Schwartz dynamics in (4.11). Thus, the critique of the dividend interpretation of the Gibson-Schwartz convenience yield is not a criticism of the mathematics per se; but rather a critique of the economic motivation for the mathematics and its connection to real-world economics. In contrast, the Pilipovic/Schwartz-Smith model does not suggest that commodity prices have anything to do with asset prices. Rather, it simply starts with the premise that RN commodity price dynamics have persistent and transitory components qualitatively like those in the objective price process.

This model is easily adapted to include multiple transitory factors:

$$s_t = L_t e^{\epsilon_{1t} + \epsilon_{2t}},$$

$$dL_t = \mu_{RN} L_t \ dt + \sigma L_t \ dZ_{L,t},$$

$$d\epsilon_{it} = \alpha_i [\theta_i - \epsilon_{it}] \ dt + \xi_i \ dZ_{i,t}, \quad i = 1, 2$$

with the speeds of mean reversion operating on different time scales, $\alpha_1 \neq \alpha_2$. For example, $\epsilon_{1t}$ may be a rapidly mean-reverting process due to temporary equipment failure while $\epsilon_{2t}$ could be a somewhat slower mean-reverting process due to weather.
4.2. Spot-price Evolution Models

One limitation of the Pilipovic/Schwartz-Smith model is that it is not automatically consistent with an arbitrary initial market futures curve. To the extent that it implies that current market futures are inconsistent with each other, we say that the model is not arbitrage-free.\(^7\) However, the model can be made arbitrage-free if it is modified to include a deterministic time-dependent drift. For example, if the mean-reverting drift pulls the transitory component towards different levels \(\theta(t)\) over time:

\[
\begin{align*}
  s_t &= L_t e^{\epsilon_t}, \\
  dL_t &= \mu^{RN} L_t \, dt + \sigma L_t \, dZ_{L,t}, \\
  d\epsilon_t &= \alpha [\theta(t) - \epsilon_t] \, dt + \xi \, dZ_{\epsilon,t},
\end{align*}
\]

then \(\theta(t)\) can be calibrated to ensure that the RN expected future spot prices agree with any current market futures price term structure.

Jaillet-Ronn-Tompaidis: This is a one-factor mean-reverting RN spot price model with a deterministic seasonal price level:

\[
\begin{align*}
  s_t &= g(t) e^{\epsilon_t}, \\
  d\epsilon_t &= \alpha [\theta - \epsilon_t] \, dt + \sigma \, dZ_t.
\end{align*}
\]

In this specification there are no persistent random shocks to prices. All randomness comes from short-term temporary deviations from a seasonally-adjusted normal price \(g(t)\). The associated futures prices are given by

\[
F_{t,T} = g(t) \exp \left\{ e^{-\alpha(T-t)} \epsilon_t + \left[ 1 - e^{-\alpha(T-t)} \right] \theta + \left[ 1 - e^{-2\alpha(T-t)} \right] \sigma^2 \frac{t}{4\alpha} \right\}.
\]

The time-dependent scaling function \(g(t)\) allows the model to be calibrated to match the current futures curve.

Stochastic volatility: An alternative to price- and time-dependent heteroskedasticity are spot price models with a stochastic volatility

\(^7\)Since Gibson-Schwartz is mathematically equivalent to Pilipovic/Schwartz-Smith, the same limitation applies there too.
factor as in

\[ ds_t = \alpha \left[ \theta_s(t) - s_t \right] dt + v(s_t) s_t dZ_{s,t}, \quad (4.20) \]
\[ dx_t = \kappa \left[ \theta_x(t) - x_t \right] dt + \sigma dZ_{x,t}. \]

The spot prices here are mean-reverting with seasonal mean price levels and the local volatility is a function of one (or possibly more) random volatility factors \( x_t \). The volatility factor could reflect randomly changing slopes of the supply and demand curves or randomly changing underlying weather or microeconomic factor volatilities. The volatility factor mean-reverts to a time-dependent level \( \theta_x(t) \) at a speed \( \kappa \) with a volatility of volatility ("vol vol") of \( \sigma \). The time-dependent price level \( \theta_s(t) \) lets the process match any futures price term structure and the seasonal volatility level \( \theta_x(t) \) lets the process exhibit a wide range of volatility term structures.

**Mean-reverting jump diffusion (MRJD):** Jumps are an important part of daily (and intra-day) power price dynamics. For example, Figure 4.8 shows how jumps in electricity prices can occur. Power plant or transmission failures cause the aggregate short-term supply curve to shift back to the left and, thereby, make the intersection with the short-run demand curve jump discontinuously. The more inelastic (steeper) the demand curve is, the larger the price jumps are.

Over time, as the grid operator finds new sources of power, as repair crews fix broken equipment, and as price-sensitive users curtail their power use, the supply and demand curves adjust causing prices eventually to mean-revert back towards the old seasonal equilibrium price and quantity.

The MRJD diffusion model captures these various qualitative features:

\[ ds_t = \alpha \left[ \theta(t) - s_t \right] dt + v(s_t, t) dZ_t + \xi(s_t, t) dJ_t. \quad (4.21) \]

Prices wander around a seasonal price level \( \theta(t) \) driven by heteroskedastic Brownian Motion shocks and Poisson jumps with state-dependent jump intensities \( \lambda(s_t, t) \) and jump magnitudes \( \xi(s_t, t) \).
4.3 Futures Term Structure Models

Another widely-used approach for commodity option valuation directly models the RN dynamics of the entire term structure of futures prices. There are two general points to make here. The first is that the dynamics of the futures curve represents the process of arriving new information and the intertemporal linkages between supply and demand at different delivery dates. Different types of information can have different impacts on spot price expectations for different future delivery dates. Some events cause persistent shifts to future supply and demand over time which cause the whole term structure of future prices to shift up or down (e.g., development of new technologies like hydraulic fracting or increased construction of natural gas power plants). Other events have supply and demand effects which are more transitory and, thus, move the spot price and short-dated futures prices more than longer-dated futures prices (e.g., short-run weather patterns).

The second general point is that futures curve dynamics imply spot price dynamics and vice versa. For example, if futures prices for early delivery dates shift more than long-dated futures prices, that implies that the impact of that particular piece of news is transitory and that its effect on spot prices will predictably mean-revert away. Conversely, if a shock to spot prices is expected to mean-revert away over time,
then that predictable decay should be reflected in longer-dated futures prices reacting less than short-dated futures prices. Although the two approaches are isomorphic, the term structure modeling approach has some tractability advantages over the spot price modeling approach. In practice, the drift of the RN spot price process must typically include calibrated time-dependent deterministic functions – for example, \( \theta(t) \) in the modified Pilipovic/Schwartz-Smith model (4.17) – to be arbitrage-free given the initial futures curve observed in the market. In contrast, since futures term structure models are specified in terms of futures price changes, setting the initial term structure equal to the observed current market futures curve automatically ensures that the future RN futures prices are consistent with the current futures curve.

The modeling of correlated futures curve dynamics draws heavily on modeling techniques first developed in an interest rate context. Heath, Jarrow, and Morton (HJM) [115] derived an arbitrage-free model of the risk-neutral dynamics for a vector of instantaneous forward rates. Their key insight is that the RN drifts of forward rates are pinned down by the forward rate volatility function and the fact that forward rates must be consistent with RN expected bond returns being equal to spot interest rates. Brace, Gatarek, and Musiela [31] and Jamshidian [122] derived a version of HJM – called the *market model* – which is specified not in terms of instantaneous forward rates but in terms of a collection of discrete-term forward rates. This modification was motivated by the fact that, in practice, fixed income traders work with vectors of discrete forward rates. Kennedy [132, 131], Goldstein [107], and Longstaff, Santa-Clara and Schwartz [149] developed and work with string models which have the same dimensionality as the set of forward rates. This lets the forward rate model be consistent both with the initial futures curve and with the empirical correlation structure of forward rates.

There is, however, at least one significant difference between modeling term structures of forward rates and modeling term structures of commodity futures prices. Futures prices are RN martingales. Thus, some of the mathematical complications associated with specifying RN forward interest rate drifts that are consistent with RN bond price dynamics do not arise in a futures price term structure model. This
4.3. Futures Term Structure Models

greatly simplifies commodity futures term structure modeling.

**Black model:** The generalized Black [19] model, discussed in §3.4, represents the RN dynamics of futures prices over time for a particular single fixed future delivery date $T$ as a driftless Ito process:

$$dF_{t,T} = v(F_{t,T}, t) F_{t,T} dZ_t.$$  \hspace{1cm} (4.22)

This can be generalized to a model of futures curve dynamics if this equation is assumed to hold for any delivery date $T$. In particular, if the standard Brownian Motion increment is not specific to a particular maturity $T$ but rather affects all futures maturities, then the Black term structure model implies that changes in futures prices for all delivery dates are locally perfectly correlated due to their common dependence on $dZ_t$. In the special case of delivery-date-independent and time- and spot price-contingent local futures price volatility,

$$dF_{t,T} = v(s_t, t) F_{t,T} dZ_t, \forall T,$$  \hspace{1cm} (4.23)

the futures curve has proportional parallel shifts. Comparing (4.23) with equation (4.6) shows that the generalized Black futures term structure model is consistent with the generalized Black-Scholes spot price model.

**Multi-factor term structure model:** The RN futures dynamics can be made richer than perfectly correlated local shifts by introducing multiple (i.e., $N$) factors and allowing futures prices for different delivery dates to load differently on the various factors:

$$dF_{t,T} = \left[ \sum_{j=1}^{N} v_j(t, x_t; T) dZ_{j,t} \right] F_{t,T}$$  \hspace{1cm} (4.24)

where the Standard Brownian Motions $dZ_{j,t}$ are cross-sectionally independent for all $j$. The factor loading notation $v_j(t, x_t; T)$ is interpreted as follows: The dependence on $T$ in $v_j(t, x_t; T)$ means that futures prices for different delivery dates $T$ along the futures curve on a given date $t$ can respond differently to a particular shock $dZ_{j,t}$. The dependence on $t$ means that the sensitivity of a futures price for a given (fixed) delivery
date $T$ to shock $dZ_{j,t}$ can change over time $t$ because of seasonalities or because factor sensitivities change as the time until delivery $T - t$ gets shorter. For example, the *Samuelson effect* – an empirical regularity that futures volatility tends to be greater for short time-to-delivery futures prices than for long time-to-delivery futures prices – is consistent with factor loadings which are decreasing in $T - t$.\(^8\) The dependence on a volatility factor $x_t$ (or, more generally, a set of factors) allows for the possibility that the factor sensitivities depend not just on time effects, but also on other random factors (e.g., the futures term structure itself or stochastic volatility factors). If $x_t$ is a non-futures state variable, then the RN dynamics for $dx_t$ must also be specified.

Cortazar and Schwartz [61] is an example of this approach. In their model, the cross-delivery-date covariances depend on both $t$ and the delivery date $T$. In the special case in which the factor loading are stable over time and just depend on maturity $T - t$, the factor loadings are often empirically calibrated using Principal Component Analysis. We can further generalize this empirical approach to estimate seasonal factor sensitivities (i.e., where the factor sensitivities depend directly on $t$) by estimating separate PCAs for each season.

In practice, a small number of factors is often sufficient for a factor model to reproduce a large part of the empirical variability of futures term structures. Litterman and Scheinkman [147] show that *level*, *slope*, and *curvature* factors explain over 90 percent of interest rate term structure variability. Secomandi, Lai, Margot, Scheller-Wolf, and Seppi [190] find that slope and curvature explain well over 90 percent of natural gas futures term structure variability.\(^9\) Working with low-dimensional factor models can sometimes improve computational tractability.

The dynamics of futures prices induce the dynamics of spot prices.

---

\(^8\) The exponential specification $dF_{t,T}/F_{t,T} = \sigma \exp[-\alpha(T - t)] dZ_t$ with $\alpha > 0$ in Clewlow and Strickland [59] is an example of such a decaying factor loading.

\(^9\) See also Cortazar and Schwartz [61], Clewlow and Strickland [59, Chapter 8], Blanco et al. [21], Tolmasky and Hindanov [209], Geman and Nguyen [100], Borovkova and Geman [28], and Frestad [96] for other empirical factor model estimations with three factors. However, Manoliu and Tompaidis [153], Borovkova and Geman [27], Suemaga et al. [200], and Wu et al. [226] use fewer than three common factors and Eydeland and Wolyniec [84, pp. 351-367], Gray and Khandelwal [109, 110], Bjerkund et al. [18], and Thompson [205, 206] use more than three common factors.
To see this, consider the log spot price at any date $t$ after the current date $t_0$:

$$\ln s_t = \ln F_{t,t} = \ln F_{t_0,t} + \int_{\tau=t_0}^{t} d\ln F_{\tau,t}. \quad (4.25)$$

From (4.24) and Ito’s Lemma, holding $t$ fixed and allowing the time index $\tau$ to vary over time between $t_0$ and $t$ gives the dynamics for futures prices $F_{\tau,t}$ with a fixed delivery date $t$:

$$d\ln F_{\tau,t} = -\frac{1}{2} \left[ \sum_{j=1}^{N} v_j^2(\tau, x_\tau; t) \right] d\tau + \sum_{j=1}^{N} v_j(\tau, x_\tau; t) dZ_{j,\tau}. \quad (4.26)$$

Substituting (4.26) into (4.25) gives

$$\ln s_t = \ln F_{t_0,t} - \frac{1}{2} \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} v_j^2(\tau, x_\tau; t) d\tau \right] + \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} v_j(\tau, x_\tau; t) dZ_{j,\tau} \right]. \quad (4.27)$$

Applying Ito’s lemma again, now allowing the time index $t$ to change, gives a stochastic differential equation for (log) spot prices:

$$d\ln s_t = \left\{ \frac{\partial \ln F_{t_0,t}}{\partial t} - \frac{1}{2} \sum_{j=1}^{N} v_j^2(t, x_t; t) \right\} dt - \frac{1}{2} \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} \frac{\partial v_j^2(\tau, x_\tau; t)}{\partial t} d\tau \right] + \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} \frac{\partial v_j(\tau, x_\tau; t)}{\partial t} dZ_{j,\tau} \right] dt + \sum_{j=1}^{N} v_j(t, x_t; t) dZ_{j,t}. \quad (4.28)$$
The four components in the drift can be interpreted as follows:\textsuperscript{10} First, the slope of the past futures curve observed on date $t_0$ for the delivery date $t$ reflects predictable (e.g., seasonal) changes in the spot price at date $t$. The next two terms in the spot price drift are Ito adjustments (related to Jensen’s inequality) due to the log transformation. The fourth term in the drift reflects any predictable autocorrelation in how spot prices respond to factor shocks over time. At date $t$ the spot price $s_t$ reflects the impact of factor shocks on prior dates $\tau$ between dates $t_0$ and $t$ as reflected in changes in prior futures prices $F_{\tau,t}$ with delivery at time $t$. Similarly, on a date $t + \Delta t$ (for a small $\Delta t > 0$) the spot price $s_{t+\Delta t}$ also reflects those same past factor shocks on dates $\tau$ but as reflected in the changes in the prior futures prices $F_{\tau,t+\Delta t}$ with delivery at $t + \Delta t$. The derivative $\partial v_j(\tau, x_{\tau}; t)/\partial t$ represents the differential impact of those past shocks on the date $t + \Delta t$ delivery futures prices relative to the impact on the date $t$ delivery futures prices in the limit as $\Delta t \to 0$. To the extent that the integrals in the third and fourth terms in the drift depend on the realized sequence of factor values $x_{\tau}$ (which includes possible dependence on the path of futures prices and/or spot prices) and Standard Brownian Motion shocks $dZ_{j,\tau}$ over time, the spot price process will be non-Markovian.\textsuperscript{11}

\textbf{String models:} If the number of factors is smaller than the dimension of the variance-covariance matrix of the futures curve, then the factor model imposes restrictions on the futures curve variance-covariance matrix since the rank of this matrix is less than the number of futures prices in the futures curve. In practice, option traders sometimes want models which allow for some amount of independent variation at each point along the futures curve. The string model approach – drawing on Kennedy [132, 131], Goldstein [107], and Longstaff, Santa-Clara, and Schwartz [149] – allows for imperfectly correlated variability along the full term structure:

\begin{equation}
   dF_{t,T} = v(t, x_{t}; T) F_{t,T} dZ_{t,T}.
\end{equation}

\textsuperscript{10}Spot price changes $d \ln s_t$ can depend on current and past values of the factor $x_t$, but they do not include contemporaneous $dx_t$ shocks just like the spot price changes $ds_t$ in a stochastic volatility model (see (4.20)) do not include volatility shocks $d\sigma_t$.

\textsuperscript{11}See Carverhill [46] who gives sufficient conditions for Markovian dynamics.
This model looks similar to the proportional parallel shift Black term structure model (4.23) except that here there is a separate standard Brownian Motion, $dZ_{t,T}$, corresponding to each maturity $T$ along the futures curve as opposed to a single Brownian motion $dZ_t$ driving the entire curve. However, rather than independent Brownian Motions driving different futures prices along the term structure, the different Brownian Motions in the string model are potentially correlated. One empirical advantage of a string model is that the correlation structure of factors in a string model for $N$ futures prices and constant correlations is exactly identified by the empirical correlation matrix. In particular, there are $(N-1)N/2$ correlations to be specified in the string model and the same number of distinct correlations in the empirical correlation matrix. In contrast, a factor model with $N$ factors has $N^2$ factor loadings. Consequently, additional structure must be assumed to identify an $N$ factor model.

4.4 Equilibrium Models

Spot price evolution models and futures term structure models represent the reduced-form statistical properties of commodity price dynamics. The focus in much of “reduced form” modeling is to develop flexible forms which can be used to price real and financial commodity derivatives. Along side this work, a large body of research explicitly models the underlying economic drivers of commodity supply and demand and their impact on the equilibrium properties of commodity prices.

The early seminal work on equilibrium commodity pricing was by Kaldor [124], Working [222, 223], Telser [204], and Samuelson [179]. Later developments on commodity pricing are in Wright and Williams [224], Williams and Wright [221], Brennan [34], Deaton and Laroque [69, 70], Litzenberger and Rabinowitz [148], Routledge, Spatt, and Seppi [178], and Tayur and Yang [202] among others. Much of this work goes under the name theory of storage because of its focus on the dynamics of storage and how storage moderates the price impact of transitory – random or seasonal – shocks to commodity production and consumption.

The key idea in the theory of storage is that physical storage gives
speculators an American-style timing option of when to use the commodity. The result is that prices and aggregate storage inventories are jointly determined. Buying by speculators at date $t$ raises spot prices at date $t$ (since less is available for current consumption) and lowers expected future prices at later dates $t' > t$ (as future inventory withdrawals augment future production). The price impact of inventory affects the profitability of inventory at date $t$. In a competitive equilibrium, inventory at date $t$ adjusts the spot price at date $t$ and the probability distribution of future spot prices until either (i) speculators are indifferent about acquiring the final marginal unit of inventory or (ii) physical constraints on aggregate inventory levels or inventory changes (of the type considered in Chapter 5) bind. In particular, one important constraint is that inventory cannot be negative. When inventory hits zero, this is called a stock-out. In a stock-out, the futures curve will be backwardated. Even though investors would like to withdraw more inventory to sell even more of the commodity in the market (i.e., if current spot prices are above expected future spot prices), they are unable to do so once inventory is exhausted. In addition, the futures curve at future delivery dates can also be backwardated if the probability of future stock-outs is high enough (since the current futures curve is the RN expectations of future futures curves at later dates). Thus, these equilibrium models provide a theory for backwardation. In addition, the theory of storage implies that commodity price volatility will tend to be high in stock-outs (when markets are backwardated) due to the absence or shortage of inventory to buffer demand and supply shocks.

Recent research has also modeled commodity production. Carlson, Khokher, and Titman [43] extend the theory of storage approach to the pricing and optimal extraction of an exhaustible commodity in which natural commodity deposits in the ground are viewed as a type of storage subject to extraction frictions. Casassus, Collin-Dufresne, and Routledge [48] show how lumpy irreversible investment in commodity production capacity (e.g., development of oil reserves) leads to regime-switching between different endogenous commodity price dynamics. Kogan, Livdan, and Yaron [137] show how irreversible investment in commodity production can cause non-monotonicities in the
relation between the convenience yield curve slope and volatility.

Equilibrium models of commodity prices often assume that the marginal consumers and investors in the market have risk neutral preferences with respect to commodity prices. This may be a plausible assumption for commodities which are a relatively small part of the aggregate consumption bundle. However, it is possible that oil and other major commodities directly affect market pricing. In addition, the prices of even economically small commodities can still be affected by supply and demand correlations with systematic macroeconomic factors. Recent research seeks to understand the role of preferences in commodity pricing and risk premia in commodity-linked investments. For example, motivated by dramatic changes in commodity prices in the mid 2000s, Baker and Routledge [6] show how changes in endowments and preferences can affect both the shape of the futures curve and the futures open interest in an endowment economy with heterogeneous agents, multiple consumption goods, and recursive preferences.12 Ready [172] shows that commodity production frictions increase commodity risk premia but moderate stock risk premia in a two-good long-run risk model (see Bansal and Yaron [7]).

4.5 Empirical Research on Commodity Prices

There is a large literature on empirical commodity pricing. A thorough survey of all of this research is beyond the scope of this chapter, but we do give pointers to work on several big research questions. Our focus here is on general empirical issues and on tests of specific pricing models.

One major research question concerns whether there are risk premia in commodity futures prices. A common empirical methodology to investigate this question uses predictive regressions:

\[
s_{t+n} - s_t = a_0 + a_1 [F_{t,t+n} - s_t] + \epsilon_{t+n}
\]

(4.30)

where the change in the spot price between date \( t \) and date \( t + n \)

12Recursive preferences are a departure from expected utility preferences in that the utility \( u(c_t, V_t) \) at each date \( t \) depends on both current consumption \( c_t \) and on the value \( V_t \) of expected future utility. See Epstein and Zin [83], Kreps and Porteus [140], and the survey paper by Backus, Routledge, and Zin [5].
(i.e., over \( n \) periods) is regressed on the date \( t \) futures-spot price basis, \( F_{t,t+n} - s_t \), for the futures price with delivery on date \( t + n \). The explanatory variable in the regression can be decomposed as follows:

\[
F_{t,t+n} - s_t = [F_{t,t+n} - \mathbb{E}_t^M(s_{t+n})] + [\mathbb{E}_t^M(s_{t+n}) - s_t] \tag{4.31}
\]

where \( F_{t,t+n} - \mathbb{E}_t^M(s_{t+n}) \) represents any market risk premium (or discount) in future prices set at date \( t \) relative to the market’s expected future spot price and where \( \mathbb{E}_t^M(s_{t+n}) - s_t \) is the market’s belief about the objective expected change in the spot price based on date \( t \) information.\(^{13}\) If the risk premium is non-random and if the market is informationally efficient – i.e., in that the market’s expectation \( \mathbb{E}_t^M(s_{t+n}) \) properly reflects any predictability in \( s_{t+n} \) – then the true value of the regression coefficient \( a_1 \) is 1. Thus, if the estimated slope \( \hat{a}_1 \) is statistically different from 1, that suggests that futures prices are informationally inefficient and/or that futures prices include time-varying risk premia. Fama and French [85] find that the estimated slope coefficients are positive for many (but not all) commodities, but often seem to be less than 1. These results are consistent with the existence of time-varying risk premia in futures prices. Other work on commodity risk premia is in Bessembinder [16], Bessembinder and Chan [17], and Gorton, Hayashi, and Rouwenhorst [108]. Closely related issues are the idea in Keynes [134] that commodity futures pricing is affected by changing trading imbalances between speculators and hedgers (see Dewally, Ederington, and Fernanado [75] and references therein) and, more recently, the pricing impact of the financialization of commodities (see, for example, Buyuksahin, Haigh, Harris, Overdahl, and Robe [40] and Tang and Xiong [201]).

A second question concerns the behavior of commodity price randomness. Schwartz [182] and Kogan, Livdan, and Yaron [137] study the term structure of commodity futures price volatility. Pan [165] uses the risk-neutral price distribution inferred using the Breeden and Litzenberger [33] method to document time variation in commodity price

\(^{13}\) The \( \mathbb{E}^M \) notation allows for the possibility of market inefficiency if the market’s expectations differ from the true objective expectations. Since \( \mathbb{E}^M \) refers to the market’s beliefs about objective dynamics, it should not be confused with RN expectations \( \mathbb{E}^{RN} \). In fact, since futures prices are RN expectations, the risk-premium component of the futures-spot basis can be rewritten as \( \mathbb{E}_t^{RN}(s_{t+n}) - \mathbb{E}_t^M(s_{t+n}) \).
4.6 Summary

This chapter reviewed three approaches to modeling commodity prices and valuing commodity-linked real assets and derivatives: Spot price evolution models, futures term structure models, and equilibrium asset pricing models. These models differ in their conceptual justification, complexity, and statistical fit. However, these different approaches are not mutually inconsistent. For example, each RN spot price model implies a corresponding model of futures curve dynamics and vice versa. Reduced-form models rely on replication (when commodities are assets or when the entire futures curve and its dynamics are known) to identify the RN dynamics or simply posit RN spot price dynamics based on price-of-risk assumptions. Equilibrium models derive endogenous market prices via explicit assumptions about investor preferences. Within each of these three approaches, we presented a variety of specific single- and multi-factor models of commodity prices. Empirical evidence on commodity pricing was also reviewed.

4.7 Notes

General references for stochastic calculus and option valuation are Shreve [193], Hull [119], and Duffie [81]. References for each of the specific models discussed here are given in the body of this chapter.
Seppi [192] is another high level discussion of issues in commodity price modeling.
This chapter deals with the modeling of commodity storage assets. Section 5.1 introduces the business setting and summarizes the results presented in this chapter. Section 5.2 formulates the problem of managing a commodity storage asset as a Markov Decision Process (MDP). Sections 5.3 and 5.4 study the structure of an optimal operating policy for this MDP, while the computation of such a policy is discussed in §5.5. Section 5.6 investigates the interplay between inventory trading and operational decisions. Section 5.7 concludes. Section 5.8 provides pointers to the literature.

5.1 Introduction

For a storable commodity, the economic interpretation of storage is the amount of commodity carried over to the next period from the current period; that is, storage from the prior period plus the difference between the commodity production and consumption in the current period (Williams and Wright [221]). Professional commodity storers,

\footnote{Figure 2.1 on page 17 in Chapter 2 illustrates the time evolution of the difference between production and consumption.}
hereafter referred to as merchants, trade this surplus in wholesale markets where they behave as price-takers. For example, such a setting is the natural gas market at Henry Hub, Louisiana, the delivery location of the NYMEX natural gas futures contract.

Merchants need access to storage facilities to support their commodity trading activities. They may own such facilities themselves, or hold rental contracts on their capacity. In this chapter, a storage asset refers to the facility where a commodity can be physically stored, or a contractual agreement that entitles its owner to usage of a portion of such a facility. The storage technology may take many forms, from conventional warehouses for metals and tanks for petroleum products and chemicals to underground depleted reservoirs, aquifers, and salt domes for natural gas. Storage assets feature two distinctive operational characteristics: minimum/maximum inventory levels (space) and injection/withdrawal (flow) capacity limits. On the financial side, commodity prices are notoriously variable and volatile (Chapter 4), and storage assets give merchants the real option to buy the commodity at one point in time, store it, and sell it at a later point in time to exploit intertemporal price variability and volatility.

Merchant management of a commodity storage asset requires determining an inventory trading policy that, given the current commodity spot and futures prices and the inventory in the storage facility, tells the merchant how much commodity to buy from the wholesale market and inject into this facility, or withdraw from this facility and sell into the wholesale market. This is a foundational problem in the merchant management of commodities, which has been studied in the literature on the warehouse problem. Cahn [41] introduces this problem as follows:

Given a warehouse with fixed capacity and an initial stock of a certain product, which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales?

The fixed capacity attribute that Cahn mentions refers to a finite
size warehouse without restrictions on how fast it can be filled up or emptied. In addition, storage facilities in practice often feature injection/withdrawal capacity constraints. That is, they are often capacitated in addition to having finite size. For example, this occurs in natural gas storage (see, e.g., Kobasa [136, pp. 8-9], Tek [203, Chapter 3], Maragos [154, p. 435], Geman [99, pp. 304-307]). As shown in this chapter, these additional flow capacity constraints are critical.

This chapter formulates the warehouse problem as a finite-horizon MDP (Stokey and Lucas [199], Puterman [171], Heyman and Sobel [116], and Bertsekas [14]). This formulation encompasses both the fast storage asset case and the slow storage asset case: a fast storage asset features only space constraints, while a slow storage asset features both space and capacity constraints.

In each stage, the state of the MDP includes both the merchant’s available inventory and the current commodity spot price, with this price evolving as an exogenous single-factor Markov process. In general, the state space is uncountable. This MDP optimizes the market value of the inventory trading policy of a price-taking merchant subject to space and possibly capacity limits. The presence of both space and capacity limits gives rise to kinked injection/withdrawal capacity functions of inventory. This MDP is based on risk-neutral valuation.

This chapter analyzes this real option formulation to show (i) the structure of the optimal inventory trading policy, (ii) the computation of an optimal policy, and (iii) the managerial relevance of interfacing inventory trading and operation decisions. The main results and insights of this analysis are now summarized.

The structure of the optimal inventory trading policy. The optimal policy in the fast asset case has a simple BI/DN/WS structure. In each stage, given the current price, it is optimal to trade to withdraw and sell (WS) to empty the asset or to buy and inject (BI) to fill it up, or to do nothing (DN) otherwise. This inventory policy is a critical

---

2 Modeling this price evolution with more than one factor would require including in the state the futures price term structure or more broadly other information, such as weather and operating conditions. Chapter 6 presents a stochastic dynamic programming model that includes the entire futures price term structure in its states.
level, or, equivalently, basestock level policy, in which the same type of action, BI/DN/WS, is optimal independent of the inventory level, given a stage and a spot price. In other words, the inventory affects how much must be bought or sold to fill up or empty the asset, but not when it is optimal to buy or sell. Put differently, there are two critical stage-dependent prices, a BI price and a WS price, such that in a given stage it is optimal to fill-up (empty) the storage when the current price is below (above) the BI (WS) price, and to do nothing otherwise.

In the slow asset case, a modified critical level policy, also known as a basestock target policy, is optimal. In a given stage and for a given spot price, this policy is characterized by two BI and WS basestock targets that partition the feasible inventory set into three regions, where the BI, DN, and WS actions are respectively optimal. When a BI/WS action is optimal, it is optimal to buy-and-inject/withdraw-and-sell that amount of inventory that brings the current inventory level as close as possible to the BI/WS target. That is, these targets may not be reachable from some inventory levels. The BI/DN/WS structure is more complex than in the fast asset case, because in the fast case only one of these regions is nonempty in a given stage and for a given spot price. In contrast, in a given stage, the optimal BI and WS prices of a slow asset are inventory-dependent.

The optimal basestock targets are functions of the stage and the spot price. Under some assumptions on the risk-neutral dynamics of the spot price, in every stage the optimal basestock levels/targets decrease in the spot price. In every stage, the inventory and spot price space is thus partitioned into three ordered BI, DN, and WS regions.

The computation of an optimal policy. In some cases of practical relevance, the optimal basestock levels/targets can be easily computed: the operational parameters of the storage asset must satisfy a benign technical condition and the spot price dynamics are modeled using a lattice representation. In this case, an optimal policy for the MDP formulation of the warehouse problem can be computed by solving a discrete state stochastic dynamic program (SDP) via backward dynamic programming. This SDP facilitates further numerical analysis.
The managerial relevance of interfacing inventory trading and operation decisions. The BI/DN/WS structure is complex in the slow asset case because given a commodity price and a stage the type of the merchant’s optimal action can depend on inventory availability. That is, at the same commodity price and stage the BI, DN, and WS actions can each be optimal at different inventory levels. This complexity is a direct consequence of the kinked capacity functions, where the kinks arise from the limited inventory adjustment capacity relative to the available space. Due to this limited capacity, the manager of a slow storage asset may at times optimally decide to make partial inventory adjustments. Put differently, this manager is concerned with managing “left over” space, that is, the available space minus the inventory adjustment capacity. This issue disappears in the fast asset case in which the left over space is always zero. Thus, trading at capacity is generally suboptimal with a slow asset, while it is optimal with a fast asset. Moreover, capacity underutilization can optimally occur at every inventory level for which trading (BI or WS) is optimal. In contrast, optimal capacity underutilization can only occur at a trivial level with a fast asset; that is, when it is optimal to do nothing.

It is also insightful to interpret the BI/DN/WS structure in the slow asset case in terms of high and low commodity prices. At a given decision stage a merchant managing a slow asset cannot always tell whether a given commodity price is high (and inventory should be sold) or low (and inventory should be bought) independently of the inventory availability. In other words, at a given time, the same price can be high or low at different inventory levels. This insight is markedly different from the inventory-independent price characterization in the fast asset case. This result occurs even if the merchant is risk-neutral and price-taker. Thus, nonlinearities in the operations of a commodity storage asset can fundamentally condition a merchant qualification of high and low commodity prices. The relevant nonlinearities in the storage asset operations are brought about by the interplay between the space and capacity limits that give rise to kinked capacity functions in the slow asset case.

The complex nature of the BI/DN/WS structure for a slow asset implies that the interface between inventory trading and operational
Modeling Commodity Storage Assets

decisions is in general nontrivial. These choices cannot be optimally de-
coupled by letting the inventory trader decide at what price to buy/sell
(that is, set the optimal BI and WS prices in a given stage) and the op-
erations manager inject/withdraw at capacity, as if the asset were fast.
This chapter uses natural gas data to quantify the importance of cou-
pling these decisions.\textsuperscript{3} It is shown here that mismanaging the trading
and operations interface can yield substantial value losses. Moreover,
incorrect inventory trading decisions magnify these losses when the in-
jection capacity is increased. A surprising part of these results is that
they hold even when the addition of constraining injection/withdrawal
capacity does not dramatically reduce the value of optimally managed
storage, which is due to the fact that in these cases the asset is op-
erated at similar flow rates. Thus, nonlinear capacity functions make
managing a commodity storage asset a genuinely difficult problem.

The results and insights of this chapter have relevance for the man-
agement of natural gas storage assets, as well as of storage assets for
other (partially) storable commodities, such as oil, metals, and agri-
cultural products (see §2.1 in Chapter 2). While the facilities used to
store these commodities may feature less stringent engineering restric-
tions in terms of injection/withdrawal capacities, in such cases capac-
ity limitations equivalent to those studied in this paper can arise as a
consequence of logistical and market constraints that limit how fast a
merchant can effectively fill up or empty a facility.

5.2 Model

This section describes a periodic review model where inventory trad-
ing decisions are made at \( N \) given equally spaced points in time. Set
\( I = \{0, \ldots, N - 1\} \) indexes them; that is, the \( i \)-th decision, \( i \in I \), is

\textsuperscript{3}This analysis is not based on the premise that policies used in practice to manage slow
commodity storage assets decouple inventory trading and operational choices. Indeed, the
practice-based policies benchmarked in Chapter 6 do consider the capacity constraints of
slow storage assets. Hence, the goal of the analysis conducted in §5.6 is not to estimate the
likely suboptimality of losses \emph{actually} incurred in practice, rather it is to understand in
a realistic setting the importance of considering the capacity constraints when managing
slow storage assets.
made at time $T_i$. An action is denoted as $a$: a positive action corresponds to a purchase followed by an injection, a negative action to a withdrawal followed by a sale, and zero is the do-nothing action. An injection or a withdrawal corresponding to a decision made at time $T_i$ is executed as a flow during the time interval in between times $T_i$ and $T_{i+1}$. This means that commodity purchased/sold at time $T_i$ is available/unavailable in storage at time $T_{i+1}$. For the most part, a buy-and-inject or withdraw-and-sell action will be simply referred to as an injection or a withdrawal, or, more generally, a trade. The monetary payoff of the $i$-th trade occurs at time $T_i$. This modeling timing reflects typical practice where financial payoffs are accounted for at specific points in time, even though physical operations often occur as flows over time. Moreover, if the length of the review period is set sufficiently short, this modeling set-up can closely approximate financial payoffs that occur simultaneously with the operational execution of the trades.

The storage asset has minimum and maximum inventory levels, $x$ and $\overline{x} \in \mathbb{R}_+$, with $x < \overline{x}$ ($\overline{x} > 0$ is common in energy applications). Hence, the feasible inventory set is $\mathcal{X} := [x, \overline{x}]$. There are constant injection and withdrawal capacities $C^I > 0$ and $C^W < 0$, respectively, on the maximum amount of inventory that can be injected into and withdrawn out of the facility in each review period (to be strict this applies to the absolute value of $C^W$). It is assumed that both $C^I$ and $-C^W$ belong to set $(0, \overline{x} - x]$. This implies that the storage asset features inventory-dependent injection and withdrawal capacity functions of inventory $\overline{a}(x) : \mathcal{X} \to [0, C^I \wedge (\overline{x} - x)]$ and $a(x) : \mathcal{X} \to [C^W \lor (x - \overline{x}), 0]$ defined as

$$\overline{a}(x) := C^I \wedge (\overline{x} - x),$$

(5.1)

$$a(x) := C^W \lor (x - \overline{x}),$$

(5.2)

where $\cdot \wedge \cdot \equiv \min\{\cdot, \cdot\}$ and $\cdot \lor \cdot \equiv \max\{\cdot, \cdot\}$. They express the maximum amount of commodity that can be injected and withdrawn, respectively, into and out of the facility during each review period starting from inventory level $x \in \mathcal{X}$ while keeping the inventory level in set $\mathcal{X}$ (strictly speaking this comment applies to the absolute value of $a(x)$). The fast asset case arises when $C^I = -C^W = \overline{x} - x$. 
Figure 5.1 illustrates the capacity functions in these cases. In the slow facility case, the kinks in these functions (at $x = \bar{x} - C^I$ and $x = \bar{x} - C^W$ in the injection and withdrawal cases, respectively) play a fundamental role in the analysis of the structure of the optimal trading policy carried out in §5.3. These kinks are not present in the fast facility case.

At any review time, the sets of feasible withdrawal and injection actions, respectively, with inventory level $x \in \mathcal{X}$ are $A^W(x) := [g(x), 0]$ and $A^I(x) := [0, \overline{p}(x)]$, and the set of all feasible actions is $\mathcal{A}(x) := A^W(x) \cup A^I(x)$. Figure 5.1 also illustrates the feasible inventory action set $\mathcal{C} := \{(x, a) : x \in \mathcal{X}, a \in \mathcal{A}(x)\}$, both in the fast and slow cases.

Let the commodity spot price be a random variable $\tilde{s}_t$, with $\mathcal{S}_t \subseteq \mathbb{R}_+$ the set of its possible realizations (for notational simplicity $T_i$ is abbre-
viated to $i$ when subscripting other variables).\footnote{The possibility of negative prices, which can occur in some electricity markets, is not considered here. Section §5.8 discusses papers that present models that admit negative electricity prices.} A Markovian stochastic process $\{\tilde{s}_i, i \in I\}$ models the evolution of the spot price, starting from the given initial price $s_0$ at the initial date $T_0$. Assuming a single-factor Markovian model of the spot price evolution, the spot price is a sufficient statistic for the state. The spot price evolves independently of the merchant trading decisions, which is consistent with the assumption of a price-taking merchant. Each merchant trade here is small relative to the size of the commodity market.

A trading decision $a$ at time $T_i, i \in I$, depends on the realized spot price, $s \in S_i$, and, by the restriction $a \in A(x)$, on the available inventory, $x$, at this time. The immediate reward (cash flow) associated with this decision is $r(a, s) : \mathbb{R} \times S \to \mathbb{R}$. Let $\phi^W \in (0, 1]$ and $\phi^I \geq 1$ denote commodity price adjustment factors used to model in-kind fuel costs that arise in the context of natural gas storage (e.g., fuel burned by compressors). Let $c^W$ and $c^I$ be positive constant marginal withdrawal and injection costs, respectively. Define the buy-and-inject adjusted price and the withdraw-and-sell adjusted price as $s^I := \phi^I s + c^I$ and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{payoff_function.png}
\caption{The immediate payoff function given two possible spot prices $s^1$ and $s^2$ with $s^1 \leq s^2$.}
\end{figure}
\( s^W := \phi^W s - c^W \), respectively. The immediate reward function is

\[
 r(a, s) := \begin{cases} 
 -s^W a, & \text{if } a \in \mathbb{R}_-, \\
 0, & \text{if } a = 0, \quad \forall i \in \mathcal{I}, \; s \in \mathcal{S}_i; \\
 -s^I a, & \text{if } a \in \mathbb{R}_+, 
\end{cases}
\]

Figure 5.2 illustrates this function (while not shown in this figure, this function is not necessarily positive in a withdrawal action). The reward function is kinked in the action at zero, as \( s^W \leq s^I \). In contrast to the kinks in the capacity functions, this kink does not play an important role in the analysis of §5.3, in the sense that the structural characterization of the optimal policy would persist even without this kink (that is, if the immediate payoff function were linear in the action).

A unit cost \( h \) for physically holding (but not financing) inventory is charged at each time \( T_i, i \in \mathcal{I} \), against the inventory \( x \in \mathcal{X} \) available at this time. Cash flows are discounted from time \( T_i \) back to time \( T_{i-1} \), \( i \in \mathcal{I} \setminus \{0\} \), using the deterministic risk-free discount factor \( \delta_{i-1} \in (0, 1] \). Notationally it is useful to define \( \delta_0 := 1 \).

An optimal inventory trading policy can be obtained by solving a finite horizon MDP. Set \( \mathcal{I} \) indexes the stages and the state space in stage \( i \) is \( \mathcal{X} \times \mathcal{S}_i \). Let \( A^\pi_i(x, s_i) \) be the decision rule of policy \( \pi \) in stage \( i \). Denote by \( \Pi \) the set of all the feasible policies. Let \( \tilde{x}^\pi_i \) be the random variable that denotes the inventory level available in stage \( i \) when following policy \( \pi \). The objective is to find an optimal policy for the following problem:

\[
\max_{\pi \in \Pi} \sum_{i=0}^{N-1} \left( \prod_{j=0}^{i} \delta_j \right) \mathbb{E}^{RN}_{x_0, s_0} [r(A^\pi_i(\tilde{x}_i^\pi, \tilde{s}_i), \tilde{s}_i) \mid x_0, s_0], \quad (5.3)
\]

where \( \mathbb{E}^{RN}[\cdot \mid x_0, s_0] \) denotes the risk-neutral expectation given the initial inventory \( x_0 \) and spot price \( s_0 \).

Model (5.3) can be equivalently reformulated as an SDP. Such a reformulation is useful both for deriving the structure of an optimal policy and computing such a policy. Denote by \( V_i(x, s) \) the optimal value function in stage \( i \) and state \((x, s)\). The SDP formulation of model (5.3) is

\[
V_{N-1}(x, s) := \max_{a \in A^W(x)} -s^W a - h x, \quad \forall (x, s) \in \mathcal{X} \times \mathcal{S}_i, \quad (5.4)
\]
5.3 Basestock Optimality

This section analyzes the structure of an optimal inventory trading policy at a given stage given the spot price.

It is useful to define the discount factors $\delta^{i,j} := 1$, $\forall i \in \mathcal{I} \setminus \{N - 1\}$, and $\delta^{i,j} := \delta^{i,j-1} \delta_{j-1}$, $\forall i \in \mathcal{I} \setminus \{N - 1\}$, $j \in \mathcal{I}$, with $j > i$. The following natural assumption on the expected discounted spot price holds throughout. It ensures that the optimal value function is finite in every stage and state.

**Assumption 1 (Expected discounted spot price).** It holds that $\delta^{j,k} \mathbb{E}^{RN}_{\tilde{s}_k | \tilde{s}_i = s_i} < \infty$, $\forall i \in \mathcal{I} \setminus \{N - 1\}$, $k \in \mathcal{J}$, with $k > j$, and $s_i \in \mathcal{S}_i$.

In the ensuing analysis, properties of functions should be interpreted in the weak sense. It is useful to define the continuation value functions

\[ W_{N-1}(x, s) := 0, \forall (x, s) \in \mathcal{X} \times \mathcal{S}_{N-1}, \quad (5.7) \]

\[ W_i(x, s) := \delta_i \mathbb{E}^{RN}_{\tilde{s}_{j+1} | \tilde{s}_i = s_i} [V_{j+1}(x, \tilde{s}_{j+1}) | \tilde{s}_i = s_i], \quad \forall i \in \mathcal{I} \setminus \{N - 1\}, \ (x, s) \in \mathcal{X} \times \mathcal{S}_i. \quad (5.8) \]
5.3.1 Fast Asset

In the case of a fast asset, Proposition 1 provides the structure of the optimal value and continuation value functions for a given spot price and stage.

**Proposition 1 (Linearity).** The optimal value and continuation value functions $V_i(x, s)$ and $W_i(x, s)$ are linear in inventory $x \in \mathcal{X}$ in every stage $i \in \mathcal{I}$ and for each given spot price $s \in \mathcal{S}_i$.

Proposition 1 implies that in every stage $i$ and for each given spot price $s$ the marginal value of inventory – that is, the slope of the value function, $V_i(x, s)$, with respect to inventory – is a constant, denoted by $\zeta_i(s)$. As the storage asset can be completely filled up or emptied in a single stage, the impact on the value of the asset of every additional unit of inventory is the same. Moreover, Proposition 1 implies that in every stage $i$ and for each given spot price $s$ the current expected next stage marginal value of inventory, that is, the slope of the continuation value function, $W_i(x, s)$, with respect to inventory, is a constant, denoted by $\overline{\zeta}_i(s)$, which satisfies $\overline{\zeta}_i(s) \equiv \delta \mathbb{E}^{RN}[\tilde{\zeta}_{i+1}|\tilde{s}_i = s]$ for every stage $i \in \mathcal{I} \setminus \{N - 1\}$.

The quantity $\overline{\zeta}_i(s)$ is what Charnes et al. [53] refer to as the inventory evaluator. This quantity plays an important role in characterizing the structure of an optimal policy for a fast asset, as stated in Theorem 5.1. An optimal action is denoted $a_i^*(x, s)$, and the quantities $\underline{b}_i(s)$ and $\overline{b}_i(s)$ are feasible inventory levels that are referred to as stage- and spot-price-dependent basestock levels. Below, the quantities $s_i^W$ and $s_i^I$, respectively, are the buy-and-inject and withdraw-and-sell adjusted prices in stage $i$.

**Theorem 5.1 (Optimal basestock levels).** In every stage $i \in \mathcal{I}$, given the spot price $s \in \mathcal{S}_i$, the basestock levels and optimal actions are (BI) (buy and inject) $\underline{b}_i(s) = \overline{b}_i(s) = \pi$ and $a_i^*(x, s) = x$ if $s_i^I < \overline{\zeta}_i(s)$, (DN) (do nothing) $\underline{b}_i(s) = \overline{b}_i(s) = x$, and $a_i^*(x, s) = 0$ if $s_i^W \leq \overline{\zeta}_i(s) \leq s_i^I$, and (WS) (withdraw and sell) $\underline{b}_i(s) = \overline{b}_i(s) = x$ and $a_i^*(x, s) = x - x$ if $\overline{\zeta}_i(s) < s_i^W$, $\forall i \in \mathcal{I}$. 

Figure 5.3 illustrates the result stated in Theorem 5.1 for a given stage and spot price. The determination of an optimal action clearly involves comparing the inventory evaluator, $\zeta(s_i)$, against the buy-and-inject or withdraw-and-sell adjusted prices, $s^I_i$ or $s^W_i$. In particular, it is optimal to buy-and-inject or withdraw-and-sell when the inventory evaluator exceeds the buy-and-inject spot price or when the withdraw-and-sell spot price exceeds the inventory evaluator, and to do nothing otherwise. As the inventory evaluator does not depend on the inventory level, the same type of action is optimal at all inventory levels. Consequently, it is optimal to purchase inventory to fill the asset up to inventory level $\bar{x}$ if the buy-and-inject adjusted price is less than the inventory evaluator, withdraw inventory to empty the asset down to inventory level $\underline{x}$ if the withdraw-and-sell adjusted price exceeds the inventory evaluator, and do nothing otherwise.

This is a simple BI/DN/WS structure, in which if BI is optimal the optimal basestock levels $\bar{b}_i(s)$ and $\underline{b}_i(s)$ are both equal to $\bar{x}$, if WS is optimal they are both equal to $\underline{x}$, and if DN is optimal then the BI basestock level $\bar{b}_i(s)$ and $\underline{b}_i(s)$ is equal to $\underline{x}$ and the WS basestock level $\bar{b}_i(s)$ is equal to $\bar{x}$. 
5.3.2 Slow Asset

The analysis of the slow asset case generalizes the structural results for the fast asset.

Proposition 2 gives a basic property of the optimal continuation value and value functions for slow assets.

**Proposition 2 (Concavity).** In every stage $i \in I$, the optimal continuation value function $W_i(x, s)$ and the optimal value function $V_i(x, s)$ are concave in inventory $x \in \mathcal{X}$ for each given spot price $s \in \mathcal{S}_i$.

In contrast to the fast case, Proposition 2 implies that the marginal value of inventory and the inventory valuator are no longer constant given a stage and a spot price. That is, these quantities are inventory-dependent. Due to the limited injection and/or withdrawal capacities, every additional unit of inventory has a smaller impact on the asset value when inventory increases. This difference between the fast and slow asset cases directly affects the optimal basestock levels, which, as stated in Theorem 5.2, now assume the interpretations of optimal basestock *targets*, which may not be reachable from some inventory levels.

**Theorem 5.2 (Optimal basestock targets).** In every stage $i \in I$, there exist critical inventory targets $\bar{b}_i(s)$ and $\underline{b}_i(s) \in \mathcal{X}$, with $\underline{b}_i(s) \leq \bar{b}_i(s)$, which depend on the spot price $s$, such that an optimal action in each state $(x, s) \in \mathcal{X} \times \mathcal{S}_i$ is

$$a^*_i(x, s) = \begin{cases} (\bar{b}_i(s) - x) \land C^I, & \text{if } x \in [\underline{x}, \bar{b}_i(s)), \\ 0, & \text{if } x \in [\underline{b}_i(s), \bar{b}_i(s)], \\ (\bar{b}_i(s) - x) \lor C^W, & \text{if } x \in (\bar{b}_i(s), \overline{x}]. \end{cases}$$  \hspace{1cm} (5.9)

Figure 5.4 illustrates the intuition behind the optimal basestock target results given in Theorem 5.2 for a given stage and spot price. The solid line segments in this figure are the expected next stage marginal value of inventory, which corresponds to the left derivative with respect to inventory, $\zeta_i(x, s)$, of the continuation value function $W_i(x, s)$. In
contrast to the fast asset, the expected next stage marginal value of inventory now depends on the inventory level, and hence the extension of the notation $\zeta_i(s)$ to $\zeta_i(x, s)$. The function $\zeta_i(x, s)$ is defined as follows ($\mathcal{X}^o$ is the interior of $\mathcal{X}$):

$$
\zeta_i(x, s) := \lim_{\epsilon \downarrow 0} \frac{W_i(x, s) - W_i(x - \epsilon, s)}{\epsilon}, \quad \forall (x, s) \in \mathcal{X}^o \times \mathcal{S}_i.
$$

This definition is extended to inventory levels $\underline{x}$ and $\overline{x}$ as follows:

$$
\overline{\zeta}_i(\underline{x}, s) := \lim_{x \downarrow \underline{x}} \zeta_i(x, s), \quad \forall s \in \mathcal{S}_i,
$$

$$
\overline{\zeta}_i(\overline{x}, s) := \lim_{x \uparrow \overline{x}} \zeta_i(x, s), \quad \forall s \in \mathcal{S}_i.
$$

With these conventions, $\overline{\zeta}_i(x, s)$ is defined for all $x \in \mathcal{X}$ for each $s \in \mathcal{S}_i$.

Figure 5.4 assumes that the optimal continuation value function is piecewise linear concave in inventory (Section 5.5 provides conditions for this to be the case). As in the fast case, the BI action or the WS action is optimal, respectively, if the buy-and-inject adjusted price $s_i^I$ or the withdraw-and-sell adjusted price $s_i^W$ is below or above the discounted expected next stage marginal value of inventory, $\overline{\zeta}_i(x, s_i)$, and
the DN action is optimal otherwise. However, given the spot price, since the marginal value of inventory decreases in inventory, this function brackets the buy-and-inject and the withdraw-and-sell spot prices at no more than two critical inventory levels. These critical inventory levels are the basestock targets \(b_i(s)\) and \(\bar{b}_i(s)\). The BI basestock target is less than the WS basestock target because the withdraw-and-sell adjusted price is less than the buy-and-inject spot price. These targets thus partition the feasible inventory set into no more than three contiguous regions, where the BI, DN, and WS actions are respectively optimal. These regions are the sets \([x, b_i(s)]\), \([b_i(s), \bar{b}_i(s)]\), and \((\bar{b}_i(s), \overline{x})\).

This is an extended version of the BI/DN/WS structure that holds for the fast asset case. With a slow asset, the optimal basestock targets in a given stage and for a given spot price are not necessarily equal to the minimum or maximum inventory levels. This is why they are targets rather than levels.

These observations yield the following insights. First, in the slow asset case, at the same spot price any type of action can be optimal in a given stage depending on the available inventory. Thus, in general, it is impossible to provide an absolute characterization of a spot price in a given stage as low or high; that is, one at which BI or WS is optimal, respectively. Any such statement must be made relative to inventory availability. In contrast, in the fast facility case in a given stage it is indeed possible to define a price as low, intermediate (that is, DN is optimal at this price), and high independently of the available inventory.

Second, in the fast asset case, when trading (BI or WS) is optimal, it is optimal to fully utilize the capacity functions. An optimal BI action is equal to the value taken by the injection capacity function (5.1) and an optimal WS action is equal to the value taken by the withdrawal capacity function (5.2). In contrast, in the slow asset case, when trading is optimal, it is in general optimal to underutilize the available capacity, as expressed by the capacity functions, at some inventory levels. What may be less clear is that this capacity underutilization can occur at every inventory level for which trading is optimal. Example 1 illustrates this possibility.
5.3. Basestock Optimality

Example 1 (Slow injection capacity). In this example there are three time periods \((N = 3)\). To emphasize the role played by the capacity functions (5.1)-(5.2), the injection/withdrawal marginal costs and the holding cost are zero and the price adjustment factors and the discount factor are one. Thus, it holds that \(r(a, s) = -sa, i = 0, 1, 2\). Prices are deterministic. As illustrated in Figure 5.5, the path of prices is \(s_0 = s^M, s_1 = s^L, \) and \(s_2 = s^H\), where \(s^H > s^M > s^L > 0\) and the superscripts H, M, and L abbreviate high, medium, and low, respectively.

With a fast asset, clearly, one would fill up the facility in stage one at the low price \(s^L\), and sell the entire inventory (down to \(x\)) in stage two, at the high price \(s^H\).

Now consider a storage facility that can be emptied in one period (fast withdrawal capacity function), while filling it up requires more than one period but less than two periods (slow injection capacity function). With an empty facility, in stage zero it is optimal to buy inventory and partially fill up the facility, hence optimally underutilizing the injection capacity at each inventory level where a BI trade is optimal. With a full facility, in stage zero it is optimal to partially withdraw and sell the available inventory, hence optimally underutiliz-

---

**Fig. 5.5 Price dynamics in Example 1.**
Fig. 5.6 Optimal underutilization of the capacity functions at every inventory level in stage zero in Example 1.

\[ \bar{x} + 2C^I > \bar{x} > \bar{x} + C^I, \]
which can be rewritten as

\[ C^I < \overline{x} - \underline{x} < 2C^I. \] (5.13)

In stage two, it holds that \( V_2(x, s^H) = s^H(x - \underline{x}) \) because \( A^W(x) = [\underline{x} - x, 0], \forall x \in \mathcal{X} \). In stage one, with \( x \in \mathcal{X} \) and \( a \in [\underline{x} - x, C^I \land (\overline{x} - x)] \), it is easy to verify that \( v_1(x, a, s^L) = (s^H - s^L)a + s^H(x - \underline{x}). \) Since \( v_1(x, a, s^L) \) increases in \( a \), it follows that \( a^*_1(x, s^L) = C^I \land (\overline{x} - x) \) and \( V_1(x, s^L) = (s^H - s^L)[C^I \land (\overline{x} - x)] + s^H(x - \underline{x}), \forall x \in \mathcal{X} \). Finally, consider stage zero and focus on inventory level \( \underline{x} \). For \( a \in [0, \overline{x} - \underline{x}] \), it holds that \( v_0(\underline{x}, a, s^M) = (s^H - s^M)a + (s^H - s^L)[C^I \land (\overline{x} - \underline{x} - a)] \). Since

\[
C^I \land (\overline{x} - \underline{x} - a) = \begin{cases} C^I, & \text{if } a \in [0, \overline{x} - \underline{x} - C^I), \\ \overline{x} - \underline{x} - a, & \text{if } a \in [\overline{x} - \underline{x} - C^I, \overline{x} - \underline{x}], \end{cases}
\]

it follows that

\[
v_0(\underline{x}, a, s^M) = \begin{cases} (s^H - s^M)a + (s^H - s^L)C^I, & \text{if } a \in [0, \overline{x} - \underline{x} - C^I), \\ (s^L - s^M)a + (s^H - s^L)(\overline{x} - x), & \text{if } a \in [\overline{x} - \underline{x} - C^I, \overline{x} - \underline{x}]. \end{cases}
\]

It holds that \( \overline{x} - \underline{x} - C^I =: a^*_0(\underline{x}, s^M) \in \arg \max_{a \in [0, \overline{x} - \underline{x}]} v_0(\underline{x}, a, s^M) \) and, by (5.13), \( 0 < a^*_0(\underline{x}, s^M) < C^I = \overline{a}(\underline{x}) \). Thus, in stage zero at inventory level \( \underline{x} \) it is optimal to buy and inject without fully utilizing the injection capacity. Moreover, it holds that \( \underline{a}(\overline{x}) < -C^I =: a^*_0(\overline{x}, s^M) \in \arg \max_{a \in [\underline{x} - \overline{x}, 0]} v_0(\overline{x}, a, s^M) \). Thus, in stage zero at inventory level \( \overline{x} \) it is optimal to withdraw and sell without fully utilizing the withdrawal capacity. Since \( x + a^*_0(\underline{x}, s^M) = \overline{x} - C^I \) and \( \overline{x} + a^*_0(\overline{x}, s^M) = \overline{x} - C^I \), it holds that \( b^*_0(s^M) = \overline{b}_0(s^M) = \overline{x} - C^I \) and \( a^*_0(x, s^M) = \overline{x} - x - C^I \), \( \forall x \in \mathcal{X} \). Thus, in stage zero it is optimal to buy and inject without fully utilizing the injection capacity at every inventory level in the interval \( [\underline{x}, \overline{x} - C^I] \), and it is optimal to withdraw and sell without fully utilizing the withdrawal capacity at every inventory level in the interval \( [\overline{x} - C^I, \overline{x}] \). Figure 5.6 displays the behavior of this optimal action function and relates it to those of the injection and withdrawal capacity functions.

This example illustrates a general feature of an optimal basestock target structure for a slow storage asset: limited inventory adjustment
capacity forces the manager of a storage asset to partially shift the decision to adjust the inventory level in a given stage to stages with less attractive cash flows. In other words, the complexity of an optimal basestock target structure arises from the need to manage “left over” space or inventory arising from limited capacity. In Example 1, the space left over is the difference between the available space and the injection capacity, that is, $(\bar{x} - x) - C_I$.

5.4 Price Monotonicity of the Optimal Basestock Targets

This section characterizes the behavior of the optimal basestock targets in the spot price. For simplicity of exposition, the ensuing analysis only uses the basestock target terminology, but this analysis also applies to the fast asset case, that is, when these targets are basestock levels reachable from any inventory level.

The optimal basestock targets are said to be monotonic in the spot price if they decrease in the spot price in each stage. Example 2 shows that an optimal basestock target does not always satisfy this property.

**Example 2 (Nonmonotonic optimal BI basestock target).** Let $N = 2$, $\bar{x} = 0$, and $\bar{x} = 1$. Suppose that $\phi^I = \phi^W = 1$, $c^I = 0.04$, $c^W = 0$, $\delta_1 = 1$, $h = 0$, and $C^I = -C^W = 1$. This is a fast asset, so that $V_1(x, s) = xs$ and $W_0(x, s) = x\hat{s}_{1|0}(s)$, where, with a slight abuse of notation, $\hat{s}_{1|0}(s) := E^{RN}[\hat{s}_1|\hat{s}_0 = s]$. Assume that there are four possible spot prices in stage zero, those in set $S_0 = \{0.01, 2, 9, 12\}$, and that $\hat{s}_{1|0}(s)$ is equal to 0.0149, 2.1922, 9.0375, and 11.8498 for $s = 0.01$, 2, 9, and 12, respectively. In stage zero, $b_0(s)$ and $\overline{b}_0(s)$ are optimal solutions to $\max_{y \in [0,1]}[\hat{s}_{1|0}(s) - (s + 0.04)]y$ and $\max_{y \in [0,1]}[\hat{s}_{1|0}(s) - s]y$, respectively. Thus, $b_0(s)$ is equal to 0 if the BI unit margin, $\hat{s}_{1|0}(s) - (s + 0.04)$, is negative and to 1 if this quantity is positive; and $\overline{b}_0(s)$ is equal to 0 if the WS unit margin, $\hat{s}_{1|0}(s) - s$, is negative and to 1 if this quantity is positive. Table 5.1 displays the BI and WS unit margins and their associated optimal basestock targets and type of action for all possible spot prices in stage one. This table shows that $b_0(s)$ behaves nonmonotonically in $s$.

What makes the function $b_1(s)$ nonmonotonic in $s$ in Example 2
Table 5.1 BI and WS unit margins, optimal basestock targets, and optimal type of action in stage one in Example 2.

<table>
<thead>
<tr>
<th>Function</th>
<th>0.01</th>
<th>2</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{s}_{1</td>
<td>0}(s) - (s + 0.04) )</td>
<td>-0.0351</td>
<td>0.1522</td>
<td>-0.0025</td>
</tr>
<tr>
<td>( \bar{s}_{1</td>
<td>0}(s) - s )</td>
<td>0.0049</td>
<td>0.1922</td>
<td>0.0375</td>
</tr>
<tr>
<td>( \bar{b}_0(s) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{b}_0(s) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Optimal type of action: DN BI DN WS

is the nonmonotonic behavior of the function \( \bar{s}_{2|1}(s) - s \) in \( s \). In this example, had the latter function decreased in \( s \), the function \( b_1(s) \) would have been monotonic in \( s \). This suggests that the price monotonicity of the optimal basestock targets can be established by imposing more structure on the spot price process. The following analysis imposes on the spot price process the conditions stated in Assumption 2.

**Assumption 2 (Spot price process).** For every \( i \in \mathcal{I} \setminus \{N - 1\} \), (a) the distribution function of random variable \( \bar{s}_{i+1} \) conditional on the spot price \( s \) in stage \( i \) stochastically increases in \( s \in \mathcal{S}_i \); (b) the function \( \delta_i \mathbb{E} [\phi^I \bar{s}_{i+1} | \bar{s}_i = s] - \phi^W s \) decreases in \( s \in \mathcal{S}_i \).

The condition stated in part (a) of Assumption 2 is rather natural. It implies that the expected spot price in the next stage increases in the current spot price (see, e.g., Topkis [210, Corollary 3.9.1(a)]). Part (b) of Assumption 2 is motivated by the observation made after Example 2. It implies that the discounted expected spot price in the next stage increases at a slower rate than the spot price in the current stage. The mean-reverting Jaillot-Ronn-Tompaïdis model (Chapter 4), used in §5.6, satisfies part (a) of Assumption 2 by Theorem 4 in Müller [159], but not part (b). (Indeed, the expected spot prices in stage two conditional on the stage one spot prices used in Example 2 are obtained by using a mean-reverting model.) A model that satisfies both of these conditions is Geometric Brownian Motion with appropriate drift rate. The Schwartz-Smith model (Chapter 4) uses Geometric Brownian Motion to represent long-term variations in commodity prices and Schwartz
and Smith [183] suggest that it may be an appropriate model for the valuation of long-term real options. However, part (b) of Assumption 2 is not necessary for the optimal basestock targets to be monotonic in price, as it is easy to verify that $\bar{b}_1(s)$ is monotonic in $s$ in Example 2 when $c^f$ is zero.

Theorem 5.3 formalizes the price monotonicity of the optimal basestock targets when Assumption 2 holds.

**Theorem 5.3 (Price monotonicity).** If Assumption 2 holds, then in every stage $i \in \mathcal{I}$ the optimal basestock target functions $\underline{b}_i(s)$ and $\bar{b}_i(s)$ decrease in the spot price $s \in S_i$.

To appreciate this result, focus on the slow asset case and Figure 5.4 (an analogous argument holds in the fast asset case). As the spot price $s$ increases, both the buy-and-inject and the withdraw-and-sell adjusted prices $s^I$ and $s^W$ increase and, under Assumption 2, it is possible to show that the discounted expected next stage marginal value of inventory, that is, the function $\zeta_i(y, \cdot)$, also increases. It is thus not obvious that the optimal basestock targets decrease in the spot price. However, this would occur if the functions $\zeta_i(y, s) - s^I$ and $\zeta_i(y, s) - s^W$ decreased in the spot price $s$ (for each given stage $j$ and inventory level $y$), which is the case under Assumption 2. This is the main idea in establishing Theorem 5.3.

In words, this result follows from showing that the discounted expected next stage marginal value of inventory net of its marginal acquisition cost and marginal disposal “cost” decreases in the spot price. Consequently, the optimal basestock targets decrease in the spot price, or, equivalently, the optimal amount of inventory bought and injected (respectively, withdrawn and sold) in each stage decreases (respectively, increases) in the spot price. This is consistent with the discussion of complementarity in Topkis [210, pp. 92-93].

When Assumption 2 holds, Theorem 5.3 brings to light the structure of the optimal policy illustrated in Figure 5.7: in every stage $i \in \mathcal{I}$, the optimal basestock targets $\underline{b}_i(s)$ and $\bar{b}_i(s)$ partition the state space $\mathcal{X} \times S_i$ into disjoint BI, DN, and WS regions. How these functions decrease in the spot price can differ significantly in the slow and fast facility cases,
5.5 Computation

Consider a Markov process that in each stage can only take on a finite number of values, for example using a lattice model such as in Jaillet et al. [121] (see also Luenberger [151, Chapter 15] and Hull [117, Chapter 16]). Thus, in the following, the spot price is assumed to evolve as stated in Assumption 3.

**Assumption 3 (Finite spot price sets).** In every stage \( i \in \mathcal{I} \), the spot price set \( \mathcal{S}_i \) is finite.

This assumption implies that the random next price \( \tilde{s}_{i+1} \) conditional on each spot price \( s \in \mathcal{S}_i \) in stage \( i \) has a discrete probability distribution, \( \forall i \in \mathcal{I} \setminus \{N - 1\} \).
Under Assumption 3, an optimal policy in the fast storage asset case can be computed by dynamic programming backward recursion, despite the feasible inventory set $\mathcal{X}$ being uncountable. This is so because determining the inventory evaluator in every stage and for each given spot price only requires computing the optimal value function at the two inventory levels $\underline{x}$ and $\overline{x}$ for each possible spot price in the next stage.

The remainder of this section focuses on the slow asset case. Proposition 3 gives a useful property of the optimal value function under Assumption 3.

**Proposition 3 (Piecewise linearity).** If Assumption 3 holds, then, in every stage $i \in \mathcal{I}$, the optimal value function $V_i(x, s)$ of a slow storage asset is piecewise linear and continuous in $x \in \mathcal{X}$ for each $s \in \mathcal{S}_i$.

Under Assumption 3, computing the optimal basestock targets for a slow storage asset is simplified by the capacities, $-C^W$ and $C^I$, and the inventory limits, $\underline{x}$ and $\overline{x}$, being integer multiples of some real number. In this case, Proposition 4 states two useful properties of the optimal value function and basestock targets by leveraging the property given in Proposition 3.

**Proposition 4 (Restricted capacities and inventory limits).** Suppose that Assumption 3 holds and that there exists a maximal number $Q \in \mathbb{R}_+$ such that $-C^W$, $C^I$, $\underline{x}$, and $\overline{x}$ are all integer multiples of $Q$. In this case, in every stage $i \in \mathcal{I}$ and for each given spot price $s \in \mathcal{S}_i$, (a) the optimal value function $V_i(x, s)$ can change slope in inventory $x$ only at inventory levels that are integer multiples of $Q$, and (b) the optimal basestock levels/targets $\underline{b}_i(s)$ and $\overline{b}_i(s)$ can be taken to be integer multiples of $Q$.

The properties stated in Proposition 4 are useful because one can compute the optimal basestock targets in every stage for each spot price by restricting attention to a finite number of feasible inventory levels, namely those that are multiples of the quantity $Q$, which can be interpreted as a lot size. Thus, one needs to compute the optimal
value function in each stage and for each possible spot price only for the
1 + (\overline{x} - \underline{x})/Q feasible inventory levels \underline{x}, \underline{x} + Q, \underline{x} + 2Q, \ldots, \overline{x}. This can
be easily done by optimally solving a discrete space MDP by standard
backward recursion.

Proposition 4 always holds for a fast asset by redefining \underline{x} as 0 and
\overline{x} as \overline{x} - \underline{x}, since in this case this Proposition holds with Q = \overline{x} - \underline{x}.
For a slow storage asset, The assumption on the quantities \underline{x}, \overline{x}, C_W, and C_I in Proposition 4 can be relaxed by redefining \underline{x} as 0 and \overline{x} as \overline{x} - \underline{x} and requiring that C_W, C_I, and \overline{x} - \underline{x} be integer multiples of Q.

5.6 The Value of Optimally Interfacing Operational and
Trading Decisions

This section quantifies the managerial relevance of the BI/DN/WS
structure in the slow asset case through a computational analysis in
the context of natural gas storage. The issue being investigated is the
relevance of taking the capacity limits into account when determin-
ing a trading policy; that is, of optimally interfacing operational and
inventory trading decisions.

The analysis here assumes that the natural gas price evolves as a
single factor mean-reverting process with deterministic monthly sea-
sonality factors as in the Jaillet-Ronn-Tompaidis model discussed in
\S 4.2 in Chapter 4. Recall that in this model the spot price at time t, s_t,
is the exponential of a single mean reverting factor, \epsilon_t, multiplied by a
deterministic monthly seasonality factor, g(t): s_t = g(t) \exp(\epsilon_t). That
is, the single factor, \epsilon_t, is the natural logarithm of the deseasonalized
spot price. This factor evolves in continuous time and values according
to the stochastic differential equation

\[ d\epsilon_t = \alpha(\theta - \epsilon_t)dt + \sigma dZ_t, \]

(5.14)

where \alpha > 0, \theta, and \sigma > 0, respectively, are the long term mean
reversion level, the speed of mean reversion, and the volatility of \epsilon_t, and
dZ_t is an increment to a Standard Brownian Motion. The dynamics in
(5.14) are under the risk-neutral measure. The dynamics of the single
factor under the objective measure are also mean-reverting but with a
different mean-reversion level.
Table 5.2 displays the estimates of the parameters of this model employed in this analysis. These estimates are obtained by calibrating the model parameters to daily NYMEX natural gas futures and option prices observed in February 2006 using the approach described by Jaillet et al. [121].

The minimum and maximum inventory levels $x$ and $x$ are 0 and 10, respectively. (The normalized unit of measurement of inventory should be interpreted as an appropriate mmBtu multiple, where 1 mmBtu = 1,000,000 British thermal units.) In practice, the number of days needed to fill up and empty different types of natural gas storage facilities varies considerably, between 20-250 and 10-150, respectively (FERC [88, p. 7]). The monthly injection and withdrawal capacities are thus varied from 10% to 100% of the maximum available space (10 units) in increments of 10%. This setting covers several cases of practical
interest since it corresponds to varying the number of days required to 
fill-up/empty a facility roughly between 30 and 300.

The injection fuel loss factor is set equal to 0.01, and there is no 
loss associated with withdrawals, so that $\phi^I = 1/0.99$ and $\phi^W = 1$. 
The injection fuel loss factor accounts for the natural gas used by the 
facility pump to inject natural gas into the storage facility. No pumping 
is assumed for withdrawals. The withdrawal and injection marginal 
costs $c^W$ and $c^I$ are both set equal to $0.02/\text{mmBtu}$. These values are 
realistic (Secomandi [186]). Consistent with how natural gas storage 
assets are leased in the U.S. the holding cost is zero: $h = 0/\text{mmBtu}$.

The analysis here features an MDP formulation with 24 stages 
($N = 24$) corresponding to a monthly partition of a two-year time 
period. The first stage corresponds to the beginning of March and the 
remaining ones to the beginning of each of the next 23 months. The 
evolution of the spot price during these months is represented by a 
trinomial lattice calibrated by standard methods (Jaillet et al. [121]) to 
the first 24 NYMEX natural gas futures prices observed at the end of 
2/1/2006 illustrated in Figure 5.8. Since this trinomial lattice is built
using the risk-neutral measure, these initial futures prices are the conditional expectations of the spot price in each remaining stage under the risk-neutral measure given the spot price in the first stage. Here the initial price is $8.723/mmBtu, the closing price for the March 2006 contract on 2/1/2006. The trinomial lattice models the stochastic variability around the expected spot price. The monthly discount factor is 0.9958, which corresponds to an annual risk-free interest rate of 5% with continuous discounting. Each SDP is optimally solved using the structural results in Proposition 4, whose underlying assumptions are satisfied by the setting of this section.

The goal of the ensuing analysis is to understand the importance of optimally interfacing operational and inventory trading decisions. This analysis varies the “speed” of the storage asset by using different values for the injection/withdrawal capacity scale factors (ICSFs/WCSFs), defined as $C^I/(\bar{x} - \underline{x})$ and $|C^W|/(\bar{x} - \underline{x})$, respectively. Three policies are considered:

1. The fast capacity optimal policy (FCOP) that assumes that ICSFs and WCSFs are both 100%.
2. The slow capacity optimal policy (SCOP).
3. A decoupled operations and trading policy (DOTP) that uses the FCOP buying and selling price thresholds in each stage, but where the quantities traded are restricted by the capacity constraints. In other words, this policy buys and sells whenever FCOP does so, but its actions are constrained by the capacity functions, so that it trades at capacity but it cannot always empty/fill-up the asset in a single stage. Thus, this policy mismanages the interface between operational and trading decisions in the merchant management of the storage asset.\(^5\)

The following analysis compares the relevant value functions and other quantities of interest in stage zero given the spot price in this stage and zero initial inventory. The initial value function under a given policy is refereed to as the total asset value under this policy. This total

\(^5\)The value function of this policy can be easily computed.
5.6. The Value of Optimally Interfacing Operational and Trading Decisions

value is the sum of the asset intrinsic and extrinsic values, which are defined later in this section.

It is clear that the FCOP value function exceeds that of SCOP in each stage. Of course, FCOP is only feasible if the storage asset is fast. The SCOP value function increases when more capacity is available and becomes equal to that of FCOP when ICSF and WCSF are both equal to 1.0. Figure 5.9 displays the FCOP percentage gains on the total value of the asset relative to SCOP for each relevant ICSF and WCSF combination. As expected, these gains decrease when ICSF or WCSF increase. When ICSF and WCSF are at least 0.4 and 0.5, respectively, these gains are no more than 10%. These gains increase rapidly when WCSF or ICSF decrease below 0.3. To explain these observations, Figure 5.10 displays the ratio of the FCOP and SCOP flow rates. These flow rates are computed by folding backward the expected amounts of natural gas withdrawn during each stage by FCOP and SCOP. Focus-
Fig. 5.10 Flow rate percent gains of FCOP relative to SCOP for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow.

...ing on the withdrawn natural gas is appropriate since these policies start with zero inventory and sell all the available inventory by the last stage. That is, one would obtain the same flow rates by focusing on the amounts of injected natural gas.

The high relative value of SCOP when ICSF and WCSF are at least 0.4 and 0.5, respectively, is due to the fact that in these cases the two policies have very similar flow rates, with the FCOP flow rate (which is constant for each of the considered ICSF and WCSF combinations) being within 15% of the SCOP flow rates. It is interesting to note that in these cases the SCOP flow rates do not significantly increase by adding withdrawal capacity. Instead, the SCOP flow rates are significantly lower than the FCOP flow rate when WCSF or ICSF fall below 0.3, which explains the relatively low value of SCOP in these cases.

One might be tempted to conjecture that not much value would be lost by managing the operations and trading interface as in the DOTP;
5.6. The Value of Optimally Interfacing Operational and Trading Decisions

that is, trading as if this interface were not important. However, as shown next, this conjecture is generally incorrect.

Figure 5.11 shows the percentage total asset value losses of DOTP relative to SCOP. That a loss can exceed 100% is due to the fact that the DOTP value function can be negative. This figure illustrates that decoupling operational and trading decisions can generate significant losses in the presence of capacity constraints. These losses are relatively contained, that is, below 10%, only when WCSF is equal to 1.0 and ICSF is 0.2 or larger, or when WCSF is 0.9 and ICSF is equal to 1.0. The deep losses displayed in Figure 5.11 result from purchases and injections made under the incorrect assumption that any injected natural gas could be entirely withdrawn and sold at some later stage.

This is a basic mismatch between operational and trading decisions in the presence of capacity limits. As a consequence of this mismatch,
more injection capacity is not necessarily beneficial for DOTP for a
given level of withdrawal capacity, when the latter capacity is “low.”
For example, this can occur for WCSFs between 0.1 and 0.4. In other
words, when the operations manager and the trader do not coordinate
their decisions, it can be better to have a tighter injection capacity when
the withdrawal capacity is “too low” because limited injection capacity
avoids excessive inventory build up. In contrast, a higher withdrawal
capacity appears to be always beneficial for DOTP, since it helps to
dispose of previously accumulated inventory.

5.7 Conclusions

This chapter deals with the modeling of commodity storage assets. The
analysis is based on an MDP formulation of the classical warehouse
problem that encompasses both the fast and the slow storage asset
cases. This analysis focuses on (i) the structure of an optimal policy
to this MDP, (ii) its computation, and (iii) the managerial relevance of
coordinating inventory trading and operational decisions. The findings
from this analysis are as follows: (i) an optimal policy has a basestock
level structure in the fast asset case and a basestock target structure
in the slow asset case; under some assumptions on the spot price dy-
namics, the optimal basestock levels/targets decrease in the spot price
in each given stage; (ii) when the spot price dynamics follow a discrete
space process, the optimal basestock levels in the fast case can be eas-
ily computed, while the computation of the optimal basestock targets
in the slow case requires an additional and natural assumption on the
operational parameters of the asset; (iii) coordinating inventory trad-
ning and operational decisions when managing a slow storage asset is of
substantial managerial importance.

5.8 Notes

Bellman [9] presents a deterministic-price dynamic programming for-
mulation of the warehouse problem for a fast asset. Charnes and Cooper
[52] discuss network flow formulations of this problem. Charnes et al.
[53] consider the case of stochastic prices. Rempala [173] focuses on the
5.8. Notes

The case of limited (slow) injection capacity. The MDP model presented in §5.2 is discussed by Secomandi [186].

Dreyfus [78] shows that the optimal policy of the model of Bellman [9] has a structure analogous to that discussed in §5.3.1. Charnes et al. [53] generalize the analysis of Dreyfus [78] to the case of stochastic prices. Rempala [173] establishes a basestock structure for the case of limited injection capacity. Secomandi [186] shows the optimality of the basestock target structure in the general case when both the injection and the withdrawal capacities are slow. Secomandi [187] provides an opportunity cost view of basestock optimality.

The basestock structure is related to the basestock results available in the inventory management literature (Zipkin [229], Porteus [169]), whose focus is on managing inventory to provide adequate service levels to customers in the face of demand uncertainty, rather than optimizing the inventory trading decisions of merchants that operate in wholesale commodity markets to take advantage of price fluctuations.

The analysis of the price monotonicity of the optimal basestock targets and their computation in §§5.4-5.5 is from Secomandi [186]. Secomandi et al. [190] show that it is possible to generalize Proposition 4 by dispensing of Assumption 3. That is, Secomandi et al. [190] establish that so long as the capacity limits and the minimum and maximum inventory levels are integer multiples of the lot size $Q$, then the optimal value function is piecewise linear concave in inventory with break points that are integer multiple of $Q$ even if Assumption 3 is not satisfied. It then follows that the optimal basestock targets can be taken to be integer multiples of $Q$ even without making this assumption. That is, Assumption 3 is superfluous in Proposition 4. However, this assumption is critical to make model (5.4)-(5.6) a discrete space MDP, which can be solved by backward dynamic programming.

Natural gas storage contracts can include “ratcheted” injection or withdrawal capacity specifications. That is, the injection or withdrawal capacities are step functions of inventory. This phenomenon is due to the increased difficulty of injecting (withdrawing) natural gas when the storage facility becomes fuller (emptier). Parsons [166] points out that in this case the optimal inventory trading policy can have a
more complicated structure than the basestock structure discussed in §5.3.

Charnes et al. [53] characterize the intercept and slope of both the optimal value function and the optimal continuation value function for a fast storage asset, that is, the marginal values of space and inventory in a given stage and the discounted next stage marginal values of space and inventory. For brevity, this chapter does not emphasize this characterization. Secomandi [188] generalizes this characterization to the slow storage asset case.

The analysis of the managerial relevance of coordinating inventory trading and operational decisions is from Secomandi [186]. In particular, the estimates of the parameters of the mean reverting model reported in Table 5.2 and used in this analysis are from Wang [216]. Other authors who have used such a model in the context of natural gas storage valuation include de Jong and Walet [68], Boogert and de Jong [24], and Manoliu [152]. Chen and Forsyth [57], Kaminski et al. [127], Thompson et al. [208], and Carmona and Ludkovski [45] value natural gas storage using continuous-time stochastic control methods.

Zhou et al. [227] study the storage of electricity, the price of which can be negative, and generalize the work of Charnes et al. [53]. Schneider [180] models power spot prices that can be negative. Kim and Powell [135] and Zhou et al. [228], among others, investigate the value of storage for a wind-based power generator. Lai et al. [143] consider liquefied natural gas storage by modeling the shipping, storage, and regasification of liquefied natural gas. Guigues et al. [111] develop a stochastic programming approach to manage liquefied natural gas contracts modeling storage and cancellation provisions. Jafarizadeh [120] models oil storage.

The notation used in this chapter is an adaptation of the notation used in Secomandi [186] and Lai et al. [142]. For consistency, the same notation is used in Chapter 6, which is based on Lai et al. [142]. The price to be paid for this consistency is that the notation of this monograph is not entirely consistent with either the notation of Secomandi [186] or the notation of Lai et al. [142].
This chapter investigates how commodity storage assets are managed in practice, focusing on natural gas storage. Section 6.1 motivates this study and summarizes the main results of this chapter. Section 6.2 introduces the natural gas storage valuation problem and formulates it as an exact SDP. In contrast to the storage model considered in Chapter 5, this model includes in its states the entire forward curve rather than only the spot price. This difference makes the computation of an optimal storage policy intractable. Section 6.3 introduces heuristic policies commonly used in practice. Section 6.4 presents a tractable approximate dynamic programming (ADP) model that yields yet another heuristic policy, and discusses the computations of two upper bounds on the value of an optimal policy. Section 6.6 benchmarks the performance of these heuristics policies and upper bounds on a set of realistic instances. Section 6.7 concludes. Section 6.8 gives pointers to the literature.
6.1 Introduction

In the United States, current federal regulation separates ownership of natural gas storage facilities from the control of their capacity. That is, the owners of this capacity must make it available to users in an open access fashion via storage contracts. This chapter deals with the benchmarking of methods used to value and manage these contracts in practice. That is, in this chapter the storage asset is a rental contract for the capacity of a natural gas storage facility.

Natural gas merchants manage such contracts as real options on natural gas prices, whose value derives from the intertemporal trading of natural gas allowed by storage. Organized markets such as NYMEX and ICE trade natural gas related financial instruments, including futures and options on futures (see §2.2 in Chapter 2). Practitioners can use them to price natural gas storage contracts using risk-neutral valuation techniques (Chapter 3). Valuing a storage contract entails dynamic optimization of inventory trading decisions with capacity constraints in the face of uncertain natural gas price dynamics.

As discussed in Chapter 5, stochastic dynamic programming (Stokey and Lucas [199], Puterman [171], Heyman and Sobel [116], Bertsekas [14]) is the natural approach to this valuation problem. However, this approach is tractable only when a low-dimensional spot price model is employed to describe the stochastic evolution of the price of natural gas (see Chapters 4 and 5).

Natural gas storage traders do not appear to like this low-dimensional approach, due to doubts about whether the futures and option price dynamics implied by such price models are consistent with the dynamics of the NYMEX or ICE futures and options they trade to hedge price risk. For example, in discussing the viability of using stochastic dynamic programming for natural gas storage valuation, Eydeland and Wolyniec [84, p. 367] make the following observations:

Great care must be taken when specifying and calibrating spot processes for the use in optimization, so that they are consistent with the hedging strategy to be pursued. Additionally, even for a given set of forward in-
formation, the critical surface may exhibit unstable behavior that renders it of limited use as a hedging tool.

According to the practice-based literature (Maragos [154, pp. 449-453], Eydeland and Wolyniec [84, pp. 351-367], Gray and Khandelwal [110, 109]), the preferred modeling approach for many natural gas storage traders is to model the full dynamics of the futures term structure using high-dimensional futures price evolution models, such as those discussed in Chapter 4. Unfortunately, this modeling choice makes stochastic dynamic programming computationally intractable and necessitates the use of alternative valuation approaches based on suboptimal but computationally tractable operating policies. According again to this practice-based literature, practitioners typically value natural gas storage based on such heuristic operating policies. Interestingly, commercial software products for the valuation of natural gas storage (see, e.g., FEA [87], KYOS [141]) include heuristic valuation models that can accommodate the high-dimensional futures price evolution models.

One such heuristic combines linear programming and spread option valuation methods with or without periodic reoptimization embedded in Monte Carlo simulation. We label the two resulting heuristic policies LP and RLP, where LP abbreviates linear program and the prefix R indicates reoptimization. Another heuristic is based on reoptimization of a deterministic dynamic program, which computes the intrinsic value of storage, within Monte Carlo simulation. We label this heuristic policy RI, where I abbreviates intrinsic and R stands for reoptimization.

This chapter has two goals: (i) investigate the effectiveness of these heuristics, and (ii) discuss how to improve on their performance when possible. Achieving these goals requires the availability of a good upper bound on the value of storage and the development of alternative heuristics.

An upper bound on the value of storage is obtained by using as a starting point the value function of a tractable ADP method to value a commodity, such as natural gas, storage asset given a high-dimensional model of the evolution of the forward curve. This approach is based on transforming the intractable full-dimensional storage valuation SDP
Table 6.1 Models and Policies Studied in this Paper.

<table>
<thead>
<tr>
<th>Model</th>
<th>Without Reoptimization</th>
<th>With Reoptimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>RI</td>
</tr>
<tr>
<td>LP</td>
<td>LP</td>
<td>RLP</td>
</tr>
<tr>
<td>ADP</td>
<td>ADP</td>
<td>RADP</td>
</tr>
</tbody>
</table>

into a tractable and approximate lower-dimensional SDP. The optimal policy of this low-dimensional SDP consists of two stage and price-state dependent basestock targets.¹ Use of this property speeds up the computation of an optimal policy for the low dimensional SDP.

The optimal policy and value function of the ADP model can be used within Monte Carlo simulation to compute both lower and upper bounds on the value of a storage asset. The optimal policy of the ADP model is a heuristic policy for the exact SDP. The optimal value function of the ADP is an approximation of the optimal value function of this SDP. One upper bound is computed by applying the information relaxation and duality approach developed by Brown et al. [38]. Denote this upper bound by DUB, where D and UB abbreviate dual and upper bound, respectively. A simpler approach is to compute a perfect information upper bound, labeled PIUB. PIUB serves as a benchmark for DUB. An additional lower bound is obtained by reoptimizing the ADP model within Monte Carlo simulation. Thus, one obtains two ADP-based lower bounds, the ADP and the RADP lower bounds. Table 6.1 summarizes the models and policies analyzed in this chapter.

On a set of realistic instances based on NYMEX price data and additional data available in the energy trading literature, DUB is found to be much tighter than PIUB. DUB is thus used to benchmark the practice-based heuristics and the two ADP and RADP lower bounds on these instances. The intrinsic value of storage accounts for a relatively small amount of the total value of storage, but its computation is extremely fast. The LP heuristic is also very fast and captures sig-

¹This structure extends the basestock structure discussed in §5.3 in Chapter 5. As pointed out in §6.8, an optimal policy for the full-dimensional storage asset valuation SDP is characterized by an extended version of this structure.
nificantly more value than the intrinsic value of storage, but its sub-optimality is large when compared to DUB. The ADP lower bound exhibits less suboptimality, in most cases, but higher computational requirement than these heuristics.

Reoptimization improves the valuation performance of all the policies. In almost all the instances, the RLP, RI, and RADP lower bounds are all nearly optimal when compared to DUB. Thus, DUB is a fairly tight upper bound and plays a critical role in benchmarking the various heuristics. Of course, all these reoptimized lower bounds are more expensive to compute. In particular, the computational requirement of the RADP lower bound is higher than those of the other lower bounds. Overall, the RI heuristic strikes the best compromise between computational efficiency and valuation quality on our instances. However, in some cases the RADP policy can improve on the performance of the RI policy at the expense of increased computational effort.

The results discussed in this chapter are useful for natural gas storage traders because they provide scientific validation, support, and guidance for the use of heuristic storage valuation models in practice. Moreover, these results remain substantially similar, provided that reoptimization is used, when the seasonality in the NYMEX natural gas forward curves used to obtain them is (artificially) eliminated. This finding suggests that the insight into the benefit of reoptimization has potential relevance for the merchant storage of other commodities, such as metals, oil, and petroleum products (see §2.1 in Chapter 2), whose forward curves do not exhibit the pronounced seasonality of the natural gas forward curve.

6.2 Valuation Problem and Exact SDP

This section describes the natural gas storage valuation problem and formulates an exact SDP of this problem that extends the SDP (5.4)-(5.6) considered in §5.2 in Chapter 5. A natural gas storage contract gives a merchant the right to inject, store, and withdraw natural gas

---

2 It is straightforward to modify the MDP (5.3) presented in §5.2 in Chapter 5 to include in its states the entire forward curve in addition to the spot price. For brevity, this modified MDP formulation is not given here.
at a storage facility up to given limits during a finite time horizon. The injection and withdrawal capacity limits are expressed in million British thermal units (mmBtus) per unit of time, e.g., day. There are also limits on the minimum and maximum amounts of the natural gas inventory that the merchant can hold under such a contract. There are proportional charges and fuel losses associated with injections and withdrawals.

The wholesale natural gas market in North America features about one hundred geographically dispersed markets for the commodity. NYMEX and ICE trade financial contracts associated with about forty of these markets. The most liquid market is Henry Hub, Louisiana, which is the delivery location of the NYMEX natural gas futures contract. NYMEX also trades options on this contract. Moreover, NYMEX and ICE trade basis swaps, which are financially settled forward locational price differences relative to Henry Hub. Thus, these financial instruments make practical the risk-neutral valuation of natural-gas related cash flows.

The quantity of interest is the value of a given natural gas storage contract at the time of its inception. This value depends on how the natural gas price changes over time because a merchant uses storage to support trading in the natural gas market: buying natural gas and injecting it into the storage facility at a given point in time, storing it for some time, and withdrawing it out of the facility and selling it at a later point in time. A storage contract can be valued as the discounted risk-neutral expected value of the cash flows from optimally operating it during its tenor, given its operational constraints (Chapter 3).

Of primary interest to traders is the value of the “forward” or “monthly-volatility” component of a storage contract (Maragos [154, p. 440], Eydeland and Wolyniec [84, p. 365]). The value of this component can be hedged by trading futures contracts, and corresponds to the value of the cash flows associated with making natural gas trading decisions on a monthly block basis. Thus, attention is restricted to the valuation of monthly cash flows.

The contract tenor spans $N$ futures maturities in set $I = \{0, \ldots, N - 1\}$. Inventory trading decisions are made at each maturity time $T_i$ with $i \in I$. The notation $F_{T_i, T_j}$ denotes the futures price
at time $T_i$ with maturity at time $T_j$, $\forall i, j \in I, j \geq i$; $F_{T_i,T_i}$ is the spot price at time $T_i$. With some abuse of notation, for the most part the notation $F_{T_i,T_j}$ is replaced with the alternative notation $F_{i,j}$ to simplify the exposition. The former notation is useful when dealing with continuous time dynamics of futures prices, the latter simplifies the writing of discrete time dynamic programs. Define the forward curve at time $T_i$ as $F_i := (F_{i,j}, j \in I, j \geq i), \forall i \in I$; by convention $F_N := 0$. Notice that $F_i$ includes the spot price at time $T_i$. Define the forward curve at time $T_i$ without including the spot price as $F'_i := (F_{i,j}, j \in I, j > i), \forall i \in I \setminus \{N - 1\}; F'_{N-1} := 0$.

The numerical experiments in this chapter are based on a string model of futures price evolution (see §4.3 in Chapter 4). This approach is representative of the high-dimensional forward models discussed in the practice-based literature. The risk-neutral dynamics of the natural gas futures prices associated with each maturity date $T_i$ are described by a driftless Geometric Brownian Motion, with maturity-specific constant volatility $\sigma_i > 0$ and Standard Brownian Motion increment $dZ_{i,t}$. Moreover, the Standard Brownian Motion increments corresponding to two different maturity dates $T_i$ and $T_j$ are instantaneously correlated with constant correlation coefficient $\rho_{i,j} \in (-1, 1)$, and $\rho_{i,i} := 1$. This model is

$$\frac{dF_{t,T_i}}{F_{t,T_i}} = \sigma_i dZ_{i,t}, \forall i \in I$$
$$dZ_{i,t}dZ_{j,t} = \rho_{i,j} dt, \forall i, j \in I, i \neq j.$$  

The storage valuation problem is formulated as an SDP by modifying the one-factor spot-price SDP (5.4)-(5.6) presented in Chapter 5. The inventory review period is monthly, so that each review time corresponds to a futures price maturity. The states in stage $i$ of SDP (5.4)-(5.6) is modified by replacing the spot price $s_i$ with the forward curve $F_i$, so that the optimal value function in stage $i$ is written as $V_i(x, F_i)$. According to the practice-based literature, natural gas storage contracts typically do not seem to include a holding cost. Thus, the unit holding cost, $h$, is set to zero. For simplicity, let the one-stage risk-free discount factor be constant across stages, and denote it by $\delta$.

The minimum inventory level is normalized to 0, $x := 0$. Thus, the
set of feasible inventory levels is \( \mathcal{X} = [0, \bar{x}] \). The feasible injection action set at each inventory level is identical to the one defined in Chapter 5. In contrast, since the minimum inventory level is zero, the feasible withdrawal action set at inventory level \( x \), \( \mathcal{A}^W(x) \), is \( [C^W \cup (-x), 0] \). The feasible action set is updated accordingly.

The modified SDP for storage valuation is

\[
V_N(x_N, F_N) := 0, \quad \forall x_N \in \mathcal{X}, \quad (6.3)
\]

\[
V_i(x_i, F_i) = \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta E^{RN} \left[ V_{i+1}(x_i + a, \tilde{F}_{i+1})|F_i' \right],
\]

\[\forall i \in \mathcal{I}, \quad (x_i, F_i) \in \mathcal{X} \times \mathbb{R}^{N-i}_+, \quad (6.4)\]

where as in Chapter 5 the function \( r(a, s_i) \) is the immediate payoff of action \( a \) given the spot price \( s_i \), and where expectation \( E^{RN} \) in (6.4) is taken with respect to the risk-neutral distribution of random vector \( \tilde{F}_{i+1} \) conditional on \( F_i \), which is sufficient to evaluate this expectation under model (6.1)-(6.2). In the remainder of this chapter, and consistent with the notation used in Chapter 5, a tilde on top of a symbol denotes a random entity.

In practice, the number of maturities \( N \) associated with natural gas storage contracts is at least twelve, so that model (6.3)-(6.4) is computationally intractable because of its high-dimensional state space.

### 6.3 Practice-based Heuristics

This section describes the practice-based policies LP and I, and their reoptimization versions RLP and RI.

#### 6.3.1 Model Based on Spread Options

The LP policy is based on spread option valuation and linear programming (as explained below, it also includes a spot sale of inventory at time \( T_0 \)). A spread option is an option on the difference between two prices with a positive strike price (see §2.4). The LP policy uses spread options on the difference between two futures prices \( F_{i,j} \) and \( F_{i,i} \) on a future date \( T_i \), with \( i < j \), adjusted for the time value of money and fuel, and strike price equal to the sum of the values of the time \( T_i \) injection and withdrawal marginal costs. Such an option is referred to as
6.3. Practice-based Heuristics

the \( i-j \) spread option. Its time \( T_0 \) value is

\[
\delta^i \mathbb{E}^{RN} \left[ \left\{ \delta^{j-i} \phi^W F_{i,j} - \phi^I F_{i,i} - \left( \delta^{j-i} c^W + c^I \right) \right\}^+ | F_{0,i}, F_{0,j} \right],
\]

and is denoted by \( S_{0}^{i,j}(F_{0,i}, F_{0,j}) \). This is the time \( T_0 \) value of injecting one unit of natural gas at time \( T_i \) and withdrawing it at time \( T_j \) provided that the value of this trade is nonnegative at time \( T_i \) \( (\{\cdot\}^+ := \max\{\cdot, 0\}) \).

The LP policy works with portfolios of spread options \( \{q_{i,j}, i, j \in \mathcal{I}, i < j\} \) where \( q_{i,j} \) is the notional amount of natural gas associated with spread option \( i-j \). Such a portfolio includes notional amounts for spread options whose injections and withdrawals are associated with maturities \( 0, 1, \ldots, N - 2 \), and \( 1, 2, \ldots, N - 1 \), respectively. The LP policy also includes a spot sale \( y_0 \) of some or all of the inventory available at time \( T_0 \).

The initial step of the LP policy is to approximate the value of storage at time \( T_0 \) by constructing a portfolio of spread options and a spot sale as an optimal solution to the linear program (6.6)-(6.14) below. The decision variables in this linear program are the notional amounts in set \( \{q_{i,j}, i, j \in \mathcal{I}, i < j\} \); the inventory levels in set \( \{x_i, i \in \mathcal{I} \setminus \{0\} \cup \{N\}\} \); and the spot sale \( y_0 \). This linear program, which only depends on the time \( T_0 \) information set \( \{x_0, F_0\} \), is

\[
U_0^{LP}(x_0, F_0) := \max_{y_0,q,x} s_0 y_0 + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}, i < j} S_{0}^{i,j}(F_{0,i}, F_{0,j})q_{i,j} \tag{6.6}
\]

s.t. \( x_{i+1} = x_i + \sum_{j \in \mathcal{I}, j > i} q_{i,j} - y_0 1\{i = 0\} \)

\[
- \sum_{j \in \mathcal{I}, j < i} q_{j,i}, \forall i \in \mathcal{I}, \tag{6.7}
\]

\( x_i \leq \bar{x}, \forall i \in \mathcal{I} \setminus \{0\} \cup \{N\}, \tag{6.8} \)

\( \sum_{j \in \mathcal{I}, j > i} q_{i,j} \leq C^I, \forall i \in \mathcal{I} \setminus \{N-1\}, \tag{6.9} \)

\( y_0 \leq -C^W, \tag{6.10} \)

\(^3\)Typically storage contracts do not entail a positive initial inventory, but this decision variable is useful in the reoptimization version of this linear program.
Benchmarking the Practice Based Management of Commodity Storage Assets

\[ \sum_{i \in I, i<j} q_{i,j} \leq -C^W, \forall j \in I \setminus \{0\}, \quad (6.11) \]
\[ y_0 \geq 0, \quad (6.12) \]
\[ q_{i,j} \geq 0, \forall i, j \in I, i < j, \quad (6.13) \]
\[ x_i \geq 0, \forall i \in I \setminus \{0\} \cup \{N\}. \quad (6.14) \]

The objective function (6.6) is the value of the portfolio of spot sale and spread options. Constraint sets (6.7) and (6.8) express inventory balance and bounding conditions, respectively (\(1\{\cdot\}\) in (6.7) is the indicator function, which is equal to 1 if its argument is true and 0 otherwise). Constraint sets (6.9)-(6.11) enforce capacity constraints. Constraint sets (6.12)-(6.14) pose nonnegativity conditions on the decision variables. The quantity \(U_0^{LP}(x_0, F_0)\) is the maximized portfolio value.

Model (6.6)-(6.14) requires prices for the spread options that appear in the objective function (6.6). Once these prices are known, this model can be optimally solved very efficiently. These prices can be obtained in two ways: from (i) market quotes or (ii) a model. Market prices are not available initially if a liquid market for these spread options is not active. In addition, spread option prices are not available for future stages when using the reoptimized version of model (6.6)-(6.14) within a Monte Carlo simulation of the forward curve (see 6.3.3). Hence, in these cases spread option prices must be obtained from a model. When using the string model (6.1)-(6.2), there is no closed form pricing formula. However, they can be numerically computed or, alternatively, they can be approximated using closed form formulas, such as Kirk’s formula (Carmona and Durrleman [44]), which is used in §6.6.

Let \(y_0^{LP}(x_0, F_0)\) and \(\{q_{i,j}^{LP}(x_0, F_0), i, j \in I, i < j\}\) be a portfolio that optimally solves the LP model (6.6)-(6.14). This portfolio can be used to construct a feasible policy for model (6.3)-(6.4). This is the LP policy.
To describe the LP policy, define the following quantities:

\[
q_{i,j}^{LP}(F_{i,i}, F_{i,j}) := \begin{cases} 
0, & \text{if } \delta^j - \delta^i \phi_W F_{i,j} - (\phi_I F_{i,i} + \delta^j - \delta^i c_W + c'I) \leq 0, \\
-q_{i,j}^{LP}(x_0, F_0), & \text{otherwise.}
\end{cases}
\]  

(6.15)

These quantities depend on \(\{x_0, F_0\}\), but this dependence is suppressed from the notation for ease of exposition. Given \(F' := (F_j)'_{j=0}\), the sequence of forward curves observed up to and including time \(T_i\), the LP policy uses the quantities defined by (6.15) and the optimal spot-sale \(y_0^{LP}(x_0, F_0)\) to obtain the following feasible inventory change initiated at time \(T_i\):

\[
y_0^{LP}(x_0, F_0) 1\{i = 0\} - \sum_{j \in I, j < i} q_{j,i}^{LP}(F_{j,j}, F_{j,i}) + \sum_{j \in I, j > i} q_{i,j}^{LP}(F_{i,i}, F_{i,j}).
\]  

(6.16)

This decision does not depend on the inventory level \(x_i\). Nevertheless the LP policy is feasible for model (6.3)-(6.4), as now discussed.

Denote the time \(T_0\) value of the LP policy by \(V_0^{LP}(x_0, F_0)\), which can be estimated using Monte Carlo simulation of the forward curve evolution. Proposition 5 states that the optimal portfolio value \(U_0^{LP}(x_0, F_0)\) is no greater than the value of the LP policy \(V_0^{LP}(x_0, F_0)\), and that both of these values are lower bounds on the optimal value of storage \(V_0(x_0, F_0)\).

**Proposition 5 (LP policy value).** It holds that

\[
U_0^{LP}(x_0, F_0) \leq V_0^{LP}(x_0, F_0) \leq V_0(x_0, F_0).
\]

The first inequality in Proposition 5 follows from observing that the optimal portfolio value \(U_0^{LP}(x_0, F_0)\) underestimates the value of the LP policy \(V_0^{LP}(x_0, F_0)\), because the objective function (6.6) “double counts” the costs and fuel losses of spread options with overlapping injections and withdrawals that the LP policy combines into a single inventory adjustment. The basic idea behind the second inequality is showing that constraints (6.7)-(6.14) are sufficient for the feasibility of the LP policy for model (6.3)-(6.4).
6.3.2 Model Based on Forward Contracts

Another model of interest is the so-called intrinsic value model, which computes the value of storage that can be attributed to seasonality, as expressed by the forward curve in the initial stage. This model is the following deterministic dynamic program:

\[ U_I^T(x_N; F_0) := 0, \quad \forall x_N \in X, \quad (6.17) \]

\[ U_I^t(x_i; F_0) = \max_{a \in A(x_i)} r(a, F_{0,i}) + \delta U_{i+1}^I(x_i + a; F_0), \quad \forall i \in I, \forall x_i \in X. \quad (6.18) \]

This model computes an optimal policy that only considers the information available at time \( T_0 \). This is the I policy, which corresponds to a sequence of purchases-and-injections or withdrawals-and-sales, one for each stage, determined based on the information available at the initial time. The cash flows associated with this policy can be secured at time \( T_0 \) by transacting in the forward market for natural gas at this time. The time \( T_0 \) value of the I policy is \( U_I^0(x_0; F_0) \).

6.3.3 Models Based on Reoptimization

It is typically possible to improve the performance of the LP and I policies by reoptimizing their associated linear and dynamic programs at each maturity to take advantage of the price and inventory information that becomes available over time, implementing the action pertaining to the maturity when the reoptimization is performed, and repeating this process up to and including the last maturity. For brevity, the details of this process are not discussed here. The time \( T_0 \) values of the reoptimization versions of the LP and I policies, that is, the RLP and RI policies, are clearly lower bounds on the value of storage and can be estimated by Monte Carlo simulation of the forward curve.

6.4 ADP Model

This section discusses the ADP model, some structural results for this model, and how it can be used to compute lower and upper bounds on the value of storage.
6.4.1 ADP Policy

To reduce the computationally intractable and exact model (6.3)-(6.4) to a computationally tractable and approximate model, the approach discussed below uses information and value function approximations, which reduce the high dimensionality of model (6.3)-(6.4) in order to compute an approximate and feasible policy for this model.

The ADP model is introduced by reformulating the exact model. To this aim, define the forward curve at time $T_i$ excluding the spot and prompt month futures prices as $F''_i := (F_{i,j}, j \in I, j > i + 1), \forall i \in \mathcal{I}\{N - 2, N - 1\}; F''_{N-2} := 0$. Also define as follows the expected value function in stage $i$ and state $(x_i, s_i)$ given that the stage $i$ inventory level $x_i$ and spot price $s_i$ and the stage $i - 1$ forward curve $F''_{i-1}$ are known but the stage $i$ forward curve $F'_i$ is unknown:

$$V'_i(x_i, s_i, F''_{i-1}) := \mathbb{E}^{RN}\left[V_i(x_i, s_i, \tilde{F}'_i) \mid s_i, F''_{i-1}\right], \quad (6.19)$$

for all $i \in \mathcal{I}\{0\}$ and $(x_i, s_i, F_i) \in \mathcal{X} \times \mathbb{R}^{N-i}$. Thus, the recursion (6.4) in stage $i \in \mathcal{I}\{N - 1\}$ and state $(x_i, F_i)$ can be equivalently expressed as

$$V_i(x_i, F_i) = \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN}\left[V_{i+1}(x_{i+1} + a, \tilde{F}'_{i+1}) \mid F'_i\right],$$

$$= \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN}\left[\mathbb{E}^{RN}\left[V_{i+1}(x_{i+1} + a, \tilde{F}'_{i+1}) \mid s_{i+1} = \tilde{s}_{i+1}, F''_{i+1}\right] \mid F_{i,i+1}\right],$$

$$= \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN}\left[V_{i+1}(x_{i+1} + a, \tilde{s}_{i+1}, F''_{i+1}) \mid F_{i,i+1}\right], \quad (6.20)$$

where the second equality follows from iterated expectations, that is, breaking the one-step conditioning on $F'_i$ into an equivalent two-step conditioning on $F_{i,i+1}$ and then $F''_{i}$. The maximization in (6.20) is computationally intractable in part because of the dimensionality of

---

4 Under the string model (6.1)-(6.2), the futures price $F_{i,i+1}$ and the forward curve $F''_{i}$ are sufficient to evaluate the outer and inner expectations, respectively.
the forward curve $F''_i$, but this dimensionality can be reduced based on the following approximations:

1. **Information:** Replace $F''_i$ in (6.20) with $(F_{0,i+2}, \ldots, F_{0,N-1})$, which is information known at time $T_0$. This effectively reduces the dimensionality of maximization (6.20) because the third argument of the function $V'_{i+1}()$, $(F_{0,i+2}, \ldots, F_{0,N-1})$, is known at time $T_0$.

2. **Value function:** Replace the unknown function $V'_{i+1}()$ with an approximation. There are many functions that could be used to approximate $V'_{i+1}()$. The function used here, $U_{ADP}^{i+1}(\cdot, \cdot)$, is recursively defined as follows.

At the final stage the boundary conditions are

$$U_{ADP}^N(x_N, s_N) := 0, \forall x_N \in \mathcal{X}. \tag{6.21}$$

At earlier stages introduce the approximate value function

$$u_{ADP}^i(x_i, s_i, F_{i,i+1}) := \max_{a \in \mathcal{A}(x_i)} r(a, s_i)
+ \delta \mathbb{E}^{RN}\left[U_{ADP}^{i+1}(x_i + a, \tilde{s}_{i+1}) \mid F_{i,i+1}\right], \tag{6.22}$$

for all $i \in \mathcal{I}$ and $(x_i, s_i, F_{i,i+1}) \in \mathcal{X} \times \mathbb{R}_+^2$, and, mimicking (6.19), define

$$U_{ADP}^i(x_i, s_i) := \mathbb{E}^{RN}\left[u_{ADP}^i(x_i, s_i, \tilde{F}_{i,i+1}) \mid s_i, F_{0,i+1}\right], \tag{6.23}$$

for all $i \in \mathcal{I}$ and $(x_i, s_i) \in \mathcal{X} \times \mathbb{R}_+$, with $F_{0,N} = F_{N-1,N} := 0$.

Expressions (6.21)-(6.23) define the ADP model, which can be used to generate a feasible policy for the exact model (6.3)-(6.4) through the maximization in the right-hand side of (6.22). Call the resulting heuristic policy the ADP policy. Denote by $a_{i,ADP}(x_i, s_i, F_{i,i+1})$ the action of this policy in stage $i$ and state $(x_i, s_i, F_{i,i+1})$; if the set

$$\arg \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN}\left[U_{ADP}^{i+1}(x_i + a, \tilde{s}_{i+1}) \mid F_{i,i+1}\right], \tag{6.24}$$

is not a singleton, the action $a_{i,ADP}(x_i, s_i, F_{i,i+1})$ is set equal to the element in this set with the smallest absolute value.
Making this ADP approach computationally tractable requires (i) using finite sets for the possible values of the spot price \( s_i \) and the futures price \( F_{i,i+1} \); (ii) suitably discretizing the dynamics of this futures price into the next stage spot price; (iii) interpolating to obtain values of \( U_{i+1}^{ADP}(x_i + a, \cdot) \) that are otherwise unavailable, given the discretization performed in step (i), when computing the expectation in (6.24) using the discretized dynamics of \( F_{i,i+1} \) into \( s_{i+1} \) set in step (ii); and (iv) mapping the pair \((s_i, F_{i,i+1}) \in \mathbb{R}_+^2\) to one of the corresponding pairs obtained in step (i), when implementing the policy obtained by solving (6.24) in this manner (see Lai et al. [142] for details).

Denote the value of the ADP policy by \( V_0^{ADP}(x_0, F_0) \). This value can be estimated by Monte Carlo simulation of the forward curve, which is needed because the approximate value function computed by the ADP model (6.22)-(6.23) is not typically equal to the value function of the ADP policy when this policy is evaluated under the full information available in model (6.3)-(6.4). In other words, when implementing the ADP policy one has access to all the relevant price information, that is, the entire forward curve at a given time; when computing this policy in the ADP model only partial information is used, that is, the current spot and prompt month futures prices at a given time and the forward curve in the initial stage.

The following result holds because the ADP policy is feasible for the exact model.

**Proposition 6 (ADP policy value).** It holds that \( V_0^{ADP}(x_0, F_0) \leq V_0(x_0, F_0) \).

The computational implementation of the ADP model can benefit from the properties of the optimal value function and policy of this model. Lai et al. [142] show that the optimal policy of the ADP model has a stage and price-state dependent basestock target structure analogous to the policy established in Theorem 5.2 in Chapter 5. That is, in each stage there exist two basestock targets, which depend on the available price information, that is, the pair \((s_i, F_{i,i+1})\), such that it

---

5The function \( U_{i+1}^{ADP}(\cdot, \cdot) \) is used to obtain the ADP policy but is not the value function of this policy.
Benchmarking the Practice Based Management of Commodity Storage Assets

is optimal to buy-and-inject (respectively, withdraw-and-sell) to get as close as possible to the lower (respectively, higher) target from any inventory level below (respectively, above) such target; doing nothing is optimal at inventory levels in between these targets.

Moreover, these targets decrease in the spot price $s_i$ for each given stage $i$ and futures price $F_{i,i+1}$. This parallels the property established in Theorem 5.3 in Chapter 5. Further, suppose that the distribution of the spot price in the next stage conditional on the prompt month futures price in the current stage, denoted by $\Phi(s_{i+1}|F_{i,i+1})$, stochastically increases in the latter quantity; that is, this distribution satisfies the property that $1 - \Phi(s_{i+1}|F_{i,i+1})$ increases in $F_{i,i+1}$ for each given $s_{i+1}$. For example, this assumption is satisfied by the multidimensional Black model (6.1)-(6.2). Then, these targets increase in the futures price $F_{i,i+1}$ for each given stage $i$ and spot price $s_i$. These properties can be interpreted as follows: as the spot price increases it is optimal to purchase less and sell more; as the prompt month futures price increases it is optimal to purchase more and sell less. These natural complementarity relationships (Topkis [210, pp. 92-93]) are consistent with the monotonicity results discussed in §5.4 in Chapter 5.

6.4.2 RADP Policy

The ADP policy is computed once at time $T_0$. The reoptimization version of the ADP policy (the RADP policy) involves re-solving the ADP model in each stage after the initial stage given the information available at that time. In other words, model ADP is reoptimized at each time $T_j, j \in \mathcal{I} \setminus \{0\}$, by using the forward curve $F_j$ in place of $F_0$. Specifically, $F_{0,i+1}$ is replaced with $F_{j,i+1}$ in (6.23). The time $T_0$ value of the resulting RADP policy is typically different, and in fact higher, than that of the ADP policy. In other words, the RADP policy can typically benefit from sequential reoptimization. Since the RADP policy is feasible for the full model, its time $T_0$ value is a lower bound on $V_0(x_0, F_0)$.
6.5 Upper Bounds

The optimal value function of the ADP model can be used to compute an upper bound on the value of storage. Following Brown et al. [38], define the penalty terms

\[ p_i^{ADP}(x_i, a, s_{i+1}, F_{i,i+1}) := U_i^{ADP}(x_i + a, s_{i+1}) - \mathbb{E}^{RN}[U_{i+1}^{ADP}(x_{i+1} + a, \tilde{s}_{i+1}) | F_{i,i+1}], \]

(6.25)

for all \( i \in \mathcal{I} \) and \((x_i, a) \in \mathcal{X} \times A(x_i)\), which are based on the optimal value function of the ADP model solved at time \( T_0 \).

These penalty terms have an appealing interpretation in terms of additional value of perfect information. Consider performing action \( a \) given inventory level \( x_i \) and the prompt month futures price \( F_{i,i+1} \). If the next stage spot price \( s_{i+1} \) is known, then the value in stage \( i + 1 \) of the resulting inventory level \( x_i + a \) according to the approximate value function \( U_i^{ADP}(x_i + a, s_{i+1}) \). If this spot price is not known, the value of this inventory level is the expectation \( \mathbb{E}^{RN}[U_{i+1}^{ADP}(x_{i+1} + a, \tilde{s}_{i+1}) | F_{i,i+1}] \). The additional value of perfect information is thus the difference in the right hand side of (6.25). These quantities are used to penalize only the availability of hindsight information in the dual upper bound model discussed next; that is, they serve the purpose of “Lagrange” multipliers (duals) only on information one is not supposed to know. In other words, they are dual feasible penalties (Brown et al. [38]).

Denote by \( P_0 \) a sequence of pairs of spot and prompt-month future prices for maturities \( 0 \) through \( N - 1 \); that is, \( P_0 := ((s_i, F_{i,i+1}))_{i=0}^{N-1} \). Given \( P_0 \), solve the following dual upper bound DUB model:

\[ U_N^{DUB}(x_N; P_0) := 0, \quad \forall x_N \in \mathcal{X}, \]  
\[ U_i^{DUB}(x_i; P_0) = \max_{a \in A(x_i)} r(a, s_i) - p_i^{ADP}(x_i, a, s_{i+1}, F_{i,i+1}) + \delta U_{i+1}^{DUB}(x_{i+1} + a; P_0), \quad \forall i \in \mathcal{I}, x_i \in \mathcal{X}. \]

(6.26)  

(6.27)

This is a perfect price information model with immediate rewards pe-
nalized according to the penalty terms (6.25). Define

\[ V_DUB^0(x_0, F_0) := \mathbb{E}^{RN} \left[ U_DUB^0(x_0; \tilde{P}_0) \mid F_0 \right]. \] (6.28)

The quantity \( V_DUB^0(x_0, F_0) \) is an upper bound on the value of storage, as stated in Proposition 7, which follows from Brown et al. [38].

**Proposition 7 (DUB).** It holds that \( V_0(x_0, F_0) \leq V_DUB^0(x_0, F_0) \).

Intuitively, \( V_DUB^0(x_0, F_0) \) is an upper bound on the value of storage because it is obtained from averaging the value functions of perfect information models in which only hindsight information is penalized (that is, using terms that average to zero). The upper bound \( V_DUB^0(x_0, F_0) \) can be estimated by Monte Carlo simulation of the forward curve.

A different upper bound on the value of storage can be obtained by setting the penalty terms defined in (6.25) equal to zero and proceeding analogously to the computation of upper bound (6.28). This yields the perfect information upper bound on the value of storage, labeled PIUB (this follows from Brown et al. [38]). This upper bound provides a benchmark for the performance of the upper bound \( V_DUB^0(x_0, F_0) \).

### 6.6 Numerical Results

This section discusses the performance of the models and policies presented in §§6.3-6.4 on a set of realistic benchmark instances.

#### 6.6.1 Instances

The benchmark instances are based on market data and parameter values reported in the energy trading literature.

The two top panels of Figure 6.1 illustrate four forward curves that include the Henry Hub spot price and futures prices of the first 23 maturities (Henry Hub is the delivery location of the NYMEX natural gas futures contract). These curves were observed on four days, each corresponding to one of the four seasons of the year: 3/1/2006 (Spring), 6/1/2006 (Summer), 8/31/2006 (Fall), and 12/1/2006 (Winter). The pronounced seasonality in the natural gas forward curve is evident in these panels.
The two bottom panels of Figure 5.8 show the Black implied volatilities of the 23 futures prices on each of the four considered trading days obtained from the prices of NYMEX call options on natural gas futures. These panels indicate that futures volatilities “tend” to decrease with longer maturities, which is as expected, but also bring to light what appear to be seasonal patterns that somewhat mirror those displayed by the forward curves.

A historical correlation matrix is estimated using daily futures prices of the first 23 maturities observed between 1/2/1997 and 12/14/2006.

The one-year treasury rates on the four selected dates, as reported by the U.S. Department of Treasury, are 4.74%, 5.05%, 5.01%, and 4.87%, respectively. They are used as risk-free interest rates to generate the monthly discount factors.
Table 6.2 Absolute Values of the Injection/Withdrawal Limits (mmBtu/Month).

<table>
<thead>
<tr>
<th>Number</th>
<th>Injection</th>
<th>Withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15</td>
<td>0.30</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>0.45</td>
<td>0.90</td>
</tr>
</tbody>
</table>

The maximum inventory $\pi$ is normalized to one mmBtu. There are three pairs of injection and withdrawal limits, as shown in Table 6.2. The first capacity pair roughly reflects the capacities in Example 8.11 in Eydeland and Wolyniec [84, p. 355]. The other two capacity pairs correspond to multiplying this capacity pair by 2 and 3, respectively, to model “faster” assets. The injection and withdrawal costs are $0.02$ and $0.01$ per mmBtu, respectively, and the injection and withdrawal fuel coefficients to 1.01 and 0.99, respectively.

The distinguishing features of the benchmark instances are the number of months in the contract tenor (number of stages), the season corresponding to the initial stage (forward curve and volatilities), and the withdrawal/injection limits. The label of an instance encodes this information in the following order:

- Number of stages: 24;
- Season: Sp, Su, Fa, and Wi for Spring, Summer, Fall, and Winter, respectively;
- Injection and withdrawal pair number: 1, 2, or 3.

In total, there are twelve instances, labeled 24-Sp-1 through 24-Wi-3.

6.6.2 Results

The results discussed below are based on evaluating all the policies and the two upper bounds by starting with zero initial inventory and by simulating 10,000 futures price sample paths. Lai et al. [142] provide a detailed discussion of the computation of these policies and bounds, which is not repeated here for brevity.

The PIUB estimates are much weaker than the DUB estimates, ranging from 143% to 258% of the DUB estimates (the average standard errors of the DUB and the PIUB estimates are 0.49% and 1.32%,
Fig. 6.2 Valuation Performance of the Three Policies without Reoptimization (Percent of the DUB Value).

of the estimated dual upper bound, respectively). Thus, in the ensuing discussion, the valuation performance of each policy and its standard error are expressed as percentages of the DUB estimates.

**No reoptimization.** Figure 6.2 reports the valuation performance of the ADP, I, and LP policies, that is, the policies that do not use reoptimization. Recall that the I policy computes the intrinsic value of storage, that is, that part of the storage value that can be attributed to the seasonality in the natural gas forward curve, rather than its volatility. Thus, this policy is likely to yield lower valuations than the other policies.

Overall, the ADP and LP policies perform consistently better than
the I policy. The performance of the latter policy may be as low as 40% of the DUB value. The ADP policy performs better than the other policies on most of the instances except on the instances 24-Sp-1 and 24-Su-1, where the LP policy outperforms it.

The standard errors on the estimates of the values of the policies without reoptimization are typically around 1%. The average standard errors for the ADP, I, and LP policies are 1.18%, 1.16%, and 1.12%, respectively.

To discuss how the valuation performance of a policy depends on the injection/withdrawal capacities, define the range of valuation performances for a policy to be the difference between its minimum and maximum valuation performance figures on each of the three instances that differ only in their injection/withdrawal capacities. For example, the range of the ADP policy on instances 24-Sp-1, 24-Sp-2, and 24-Sp-3 is $(95.12 - 93.11\%) = 2.01\%$. The LP policy is the least sensitive with a rough average range of 3%, whereas the ranges of the ADP and I policies are about 4% and 8%, respectively. It appears that the ADP policy is able to capture a larger share of the value of storage for instances with higher injection/withdrawal capacities, while the intrinsic value, relative to the DUB value, becomes smaller when the injection/withdrawal capacities increase (intuitively, increasing these capacities increases the optionality of storage, and, hence, the extrinsic value of storage increases more than its intrinsic value). In particular, the values obtained by the LP policy do not show a monotone pattern as the injection/withdrawal capacities vary.

Table 6.3 reports the statistics on the Cpu times needed to compute

<table>
<thead>
<tr>
<th>Statistic</th>
<th>ADP</th>
<th>I</th>
<th>LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>146.23</td>
<td>0.53</td>
<td>0.64</td>
</tr>
<tr>
<td>Minimum</td>
<td>135.29</td>
<td>0.40</td>
<td>0.46</td>
</tr>
<tr>
<td>Average</td>
<td>140.01</td>
<td>0.46</td>
<td>0.54</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4.54</td>
<td>0.05</td>
<td>0.07</td>
</tr>
</tbody>
</table>
6.6. Numerical Results

Table 6.4 Statistics on the Cpu Seconds Needed to Compute the Three Policies with Re-optimization.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>RADP</th>
<th>RI</th>
<th>RLP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>634.22</td>
<td>24.81</td>
<td>296.12</td>
</tr>
<tr>
<td>Minimum</td>
<td>570.66</td>
<td>23.20</td>
<td>262.88</td>
</tr>
<tr>
<td>Average</td>
<td>595.73</td>
<td>23.90</td>
<td>278.74</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>20.85</td>
<td>0.47</td>
<td>12.26</td>
</tr>
</tbody>
</table>

and evaluate the different policies. The machine used for these computations is a 64 bits Monarch Empro 4-Way Tower Server with four AMD Opteron 852 2.6GHz processors, each with eight DDR-400 SDRAM of 2 GB and running Linux Fedora 9. The compiler is g++ version 4.3.0 20080428 (Red Hat 4.3.0-8). All the results are obtained using only one processor. The linear programs associated with the LP and LPN policies are solved using the Clp linear solver of COIN-OR (www.coin-or.org).

The ADP policy requires on average much more Cpu time than the other policies. However, this number can be significantly reduced by using a coarser discretization without significantly affecting the valuation performance of this policy (Lai et al. [142]). The fastest policy to compute and evaluate is the I policy, but the computational requirements of the LP policy is also very small.

Reoptimization. Figure 6.3 reports the valuation performance of the RADP, RI, and RLP policies. The RADP policy captures at least 96% of the value of storage on all the instances, except on 24-Wi-1 for which this figure is 94.91%. Moreover, the RADP policy performs mostly better than the other policies, with the exception of 24-Sp-2, where it is marginally outperformed by RLP. The performances of the RADP, RI, and RLP policies are all very insensitive to changes in injection/withdrawal capacities with average ranges less than 1%.

Similar to the case without reoptimization, the standard errors of the estimates of the values of the policies with reoptimization are around 1%. The average standard errors for the RADP, RI, and RLP
policies are 1.19%, 1.17%, and 1.17%, respectively. It is noteworthy that the gap between DUB and the value of the RADP policy is substantially higher in the Winter instances than in the instances associated with the other seasons (see panel (d) of Figure 6.3). This is due to DUB being looser on the Winter instances compared to the other instances (see the discussion in §6.8).

Table 6.4 reports the statistics on the Cpu times needed to compute the four reoptimization policies. The RADP policy needs on average substantially more Cpu time than the other policies, whose computational requirements, however, increase markedly relative to the no-reoptimization case. The policy that can be evaluated the fastest is RI; this takes roughly one tenth of the time needed to evaluate the RLP

![Fig. 6.3 Valuation Performance of the Three Policies with Reoptimization (Percent of the DUB Value).](image-url)
6.7 Conclusions

Summary. The results discussed in this section bring to light the benefit of combining reoptimization and Monte Carlo simulation for natural gas storage valuation. This approach allows one to compute a near optimal policy in a rather simple and relatively fast fashion, for example, by sequentially reoptimizing the I model, which is a deterministic and dynamic model that can be optimized very efficiently. Reoptimizing the ADP model, which is a stochastic and dynamic model, yields a slightly better policy at the expense of significantly higher Cpu requirements. Reoptimizing the LP model, which is a stochastic and static model, also generates a very good policy and is faster than reoptimizing the ADP model.

6.7 Conclusions

The valuation of natural gas storage assets is an important problem in practice. Exact valuation of these assets using the multidimensional models of the evolution of the natural gas forward curve that seem to be used in practice is an intractable problem. Thus, practitioners typically value them using heuristics. This chapter discusses an ADP approach to benchmark the effectiveness of a set of such heuristics on realistic instances and possibly improve on their valuation performance. Unlike these heuristics or methods that are available in the extant literature, this ADP approach yields both lower and upper bounds on the value of storage using the multidimensional representation of the dynamics of the natural gas forward curve that seem to be used in practice.

The estimated upper bounds appear to be fairly tight. The analyzed practice-based heuristic policies are very fast to compute but also significantly suboptimal, and are dominated by the ADP policy. However, when employed in a reoptimization fashion within Monte Carlo simulation, the valuation performances of all but one of these policies become nearly optimal. The price to be paid for this improvement is a significantly higher computational burden. Overall, sequential reoptimization within Monte Carlo simulation of a deterministic model that computes
the intrinsic value of storage strikes a good balance between valuation quality and computational requirements. However, in some cases the RADP policy can improve on the performance of the RI policy, but requires more time to compute.

These results have immediate relevance for managers of natural gas storage assets. The findings with reoptimization remain substantially similar when (artificially) removing the seasonality from the employed natural gas forward curves (Lai et al. [142]). This observation suggests that the results based on reoptimization also have potential relevance for managers involved in the valuation of storage assets for other commodities, whose forward curves do not exhibit the marked seasonality of the natural gas forward curve.

6.8 Notes

This chapter is based on Lai et al. [142]. As mentioned in §5.8 in Chapter 5, the notation used in this chapter is not fully consistent with the notation of Lai et al. [142], but it is consistent with the notation of Chapter 5.

In their study of the effect of futures price term structure model error on merchant commodity storage, Secomandi et al. [190] also investigate the structure of an optimal policy for the SDP considered in §6.2. They show that such a policy is of the basestock type, with basestock targets that in each stage depend on the entire forward curve. This structure extends the one presented in §5.3 in Chapter 5.


Bjerksund et al. [18] and Wu et al. [226] also analyze the performance of the rolling intrinsic policy. Secomandi [188] investigates structural properties of the intrinsic and basket-of-spreads policies and their rolling versions, and provides theoretical support for the near optimal
Nadarajah et al. [161] obtain stronger upper bounds for the instances considered in this chapter. They show that on the Winter instances the upper bounds used in this chapter are between 2.46% and 2.82% larger than those that they compute. This finding implies that the RI, RLP, and RADP policies are near optimal also on these instances.

The approach of Nadarajah et al. [161] is based on the approximate linear programming approach to MDPs (de Farias and Van Roy [67] and Adelman [1]) and subsumes the ADP model used in this chapter. In particular, Nadarajah et al. [161] establish that this ADP model is an approximate linear programming relaxation and propose additional approximate linear programming relaxations.

The heuristics considered in this chapter are examples of control algorithms, which are optimization-based ADP models that compute heuristic control policies for intractable MDPs (Secomandi [184]). Other ADP approaches include regression and Monte Carlo simulation (Longstaff and Schwartz [150], Bertsekas and Tsitsiklis [15]), Monte Carlo simulation and math programming (Powell [170]), rollout policies (Bertsekas [14, Chapter 6]), and Monte Carlo simulation and optimization algorithms (Chang et al. [51]). Boogert and de Jong [24, 25], Carmona and Ludkovski [45], Nascimento and Powell [164], and Nadarajah et al. [160] value natural gas storage using ADP methods based on Monte Carlo simulation. Boogert and Mazières [26] and Thompson [205] apply radial basis function techniques to value natural gas storage. Felix and Weber [90] approach the same problem using recombining trees. Desai et al. [73] develop a pathwise optimization approach to the valuation of Bermudan options. Bonnans et al. [22] develop a stochastic dual dynamic programming approach to value energy contracts.

The upper bound used in this chapter is based on the theory developed by Brown et al. [38], who generalize the work of Davis and Karatzas [66], Rogers [174], Andersen and Broadie [3], and Haugh and Kogan [114] on pricing American options (see Glasserman [103, Chapter 8] for a review). Recent work includes the papers by Belomestny et al. [10], Bender [11], Chandramouli and Haugh [50], and Meinshausen
Benchmarking the Practice Based Management of Commodity Storage Assets

and Hambly [157].
Modeling Other Commodity Conversion Assets

This brief chapter deals with the optimal management of commodity conversion assets other than storage, a topic introduced in §7.1. An inventory disposal asset is considered in §7.2, an inventory acquisition asset in §7.3, and swing assets in §7.4. Section 7.5 offers a few concluding remarks. Section 7.6 includes examples of inventory disposal and acquisition assets, in addition to the ones in §§7.2-7.3, as well as pointers to the literature.

7.1 Introduction

The commodity conversion assets considered in this chapter are related to the commodity storage assets studied in Chapter 5. Inventory disposal/acquisition assets are similar to storage assets, except that they are one-sided with only sale decisions for inventory disposal assets and only purchase decisions for inventory acquisition assets. Another difference from the storage asset is that they have also constraints on the amount of inventory that must be disposed-of/acquired by a given date. This implies that the feasible inventory set is stage dependent. Swing assets also are akin to storage assets, but their payoffs represent
savings/gains from the contractual rights to buy/sell a commodity at prices that differ from market prices.

The results of this chapter illustrate how the basestock structure that characterizes the optimal inventory trading policy for commodity storage assets can be modified for the optimal management of inventory disposal/acquisition and swing assets. Thus, the formulations of these problems as MDPs and their analysis rely substantially on the formulation of the storage asset as an MDP and the analysis carried out in Chapter 5. In addition, this chapter points out that such modified basestock structures share the same complexity of the optimal basestock structure for storage assets, as well as, under natural assumptions on the relevant operational parameters, its lot size characterization stated in part (b) of Proposition 4 in §5.5 in Chapter 5.

7.2 An Inventory Disposal Asset

Consider a merchant who has acquired a given amount of a storable commodity and wants to unwind this position in a fixed time horizon. For instance, this might be a merchant of a particular metal, such as aluminum, whose inventory is stored in a warehouse on its account (Geman [99, Chapter 8]). Sales from inventory are marked at the prevailing spot price when each sale occurs. There is a limit on the amount of commodity that the merchant can sell per unit of time, e.g., due to the warehouse operational constraints. This merchant thus owns an inventory disposal asset with an embedded timing option about when to sell.

Different from the storage asset studied in Chapter 5, for which there is no constraint on the amount of inventory to be held at the end of the planning horizon, the manager of an inventory disposal asset must sell all of the initial inventory by the end of the planning horizon. Hence, the terminal inventory level is constrained to be zero.

The problem of managing an inventory disposal asset can be formulated as an MDP. The time horizon is \([0, T]\). Inventory disposal decisions are made at each time \(T_i\), with \(i \in \mathcal{I} := \{0, \ldots, N - 1\}\), \(T_0 \equiv 0\), and \(T_{N-1} \equiv T\). The stage set is \(\mathcal{I}\). The initial inventory is \(\bar{x}\). The maximum amount of inventory that can be sold in each stage is
7.2. An Inventory Disposal Asset

the positive number $C$, which is less than or equal to $\bar{x}$. The feasible inventory set in stage $i$ is $X_i := [0, \bar{x} \land (N - i)C]$, since the initial inventory must be entirely sold off by time $T$ and the capacity $C$ limits how much inventory can be sold in each stage.

Selling an amount of inventory $a$, a nonnegative quantity, in stage $i$ generates the cash flow $(s_i - c) a$, where $c$ is a marginal operational cost. The holding cost $h$ is charged against each unit of inventory available in a given stage.

Given a feasible inventory level $x$ in stage $i$, a feasible action $a$ must be nonnegative, cannot exceed the selling capacity $C$, and must result in a feasible inventory level in the next stage, that is, a value in set $X_{i+1}$:

$$0 \leq a \leq C,$$
$$0 \leq x - a \leq \bar{x} \land ((N - i - 1)C).$$

The resulting feasible action set is

$$A_i(x) := [(x - (\bar{x} \land (N - i - 1)C)) \lor 0, x \land C].$$

For simplicity, assume that the spot price follows a one-factor process in which the current spot price is a sufficient statistic for the distribution of spot prices in later stages (see §4.2 in Chapter 4). This assumption can be easily relaxed for the purposes of the analysis performed in this section. As in §5.2 in Chapter 5, let $A_i^\pi(x, s_i)$ be the decision rule of policy $\pi$ in stage $i$, $\Pi$ the set of all the feasible policies, and $\tilde{x}_i^\pi$ the random variable that denotes the inventory level available in stage $i$ when following policy $\pi$. Denote the (constant) per stage risk-free discount factor by $\delta$ and risk-neutral expectation by $E^{RN}$.

The initial inventory level in stage 0 is $\bar{x}$. The objective is to solve the following problem:

$$\max_{\pi \in \Pi} \sum_{i=0}^{N-1} \delta^i E^{RN} [(\bar{s}_i - c) A_i^\pi(\tilde{x}_i^\pi, \bar{s}_i) - h\tilde{x}_i \mid \bar{x}, s_0]. \quad (7.1)$$

To understand the structure of the optimal policy of problem (7.1), ignore the requirement that the initial inventory must be entirely sold by time $T$. In this case the inventory disposal asset is analogous to
Modeling Other Commodity Conversion Assets

the storage asset discussed in Chapter 5 with respective injection and withdrawal capacities equal to zero and \(-C\), marginal injection and withdrawal costs equal to infinity and \(c\), and fuel loss factors equal to one. (The minus sign in front of \(C\) here is due to the withdrawal action and capacity being modeled as negative quantities in Chapter 5.) It thus follows from the analysis of Chapter 5 that the optimal policy for such an inventory disposal asset has a basestock target structure. Specifically, in every stage and for a given spot price realization, the BI basestock target can be defined to be zero and the only relevant basestock target is the WS target. In every stage and for a given spot price, the feasible inventory level is partitioned into at most two regions: a do nothing region for inventory levels below the sell-down-to basestock target and a selling region for inventory levels above this target.

This discussion suggests that adding the constraint that the entire initial inventory be sold by the end of the planning horizon should not fundamentally change the optimal policy structure. Proposition 8 formally shows that this structure is of the single basestock target type.

**Proposition 8 (Inventory disposal asset).** The optimal policy for problem (7.1) is of the basestock target type. In every stage \(i \in I\) given the spot price \(s_i \in \mathbb{R}_+\) there exists a critical inventory level \(b_i(s_i) \in X_i\) such that an optimal action in state \((x_i, s_i) \in X_i \times \mathbb{R}_+\) is \(a^*_i(x_i, s_i) = 0 \lor ((x_i - b_i(s_i)) \land C).\)

Proposition 8 relies on the basic intuition behind the basestock structure for storage assets established in Theorem 5.2 in Chapter 5. In a given stage and for a given spot price, the optimal continuation value function of the equivalent formulation of problem 7.1 as an SDP, which is not shown here for brevity, is concave in inventory. The linearity of the immediate payoff function in the action then implies the stated basestock target structure.

Analogous to storage assets, the computation of an optimal inventory disposal policy is greatly simplified by the capacity \(C\) and the maximum inventory level \(\bar{x}\) being integer multiples of a given lot size \(Q\). Indeed, in this case each optimal basestock target also can be taken to be an integer multiple of \(Q\). That is, the optimal basestock target
structure has a lot size characterization.

Example 1 in §5.3.2 in Chapter 5 shows the general complexity of the optimal basestock target structure for a slow storage asset; namely, that the optimal slow storage policy is not of the bang-bang type. The same is true for the optimal policy of a slow \((C < \bar{x})\) inventory disposal asset. This feature can be seen by tailoring this example to such an asset.

Specifically, consider the medium, low, and high deterministic spot price path illustrated in Figure 5.5 in §5.3.2 of Chapter 5. Set the total inventory to be sold \(\bar{x}\) equal to 1, the discount factor \(\delta\) equal to 1, the selling capacity \(C\) equal to 2/3, and the marginal cost \(c\) and the holding cost \(h\) to be zero. This means that the lot size \(Q\) is equal to 1/3. Given that the initial inventory level is 1, it is clearly optimal to sell 1/3 of it in stage 0 and 2/3 of it in stage 2. This means that the optimal basestock targets are 2/3 in stage 0, 1 in stage 1, and 0 in stage 2. The selling capacity is thus optimally underutilized when selling in stage 0.

Similar to Example 1 in §5.3.2 in Chapter 5 that brings to light the notion of left over space for a slow storage asset, this example illustrates that left over inventory is a general feature of an optimal basestock target structure with a slow inventory disposal asset. The inventory left over in stage 0 is 1/3, which is the difference between the initial inventory to be sold and the selling capacity per stage: \(\bar{x} - C = 1 - 2/3\).

### 7.3 An Inventory Acquisition Asset

An inventory acquisition asset is the flip-side of an inventory disposal asset. It models the situation in which a merchant has agreed to deliver a given amount of commodity by a certain date, and thus must acquire this amount during a given time horizon. In doing so, the merchant has the flexibility (timing option) of choosing when to purchase the commodity from the spot market, holding it until the delivery date.

This problem can be formulated as an MDP. The time horizon is \([0, T]\). The merchant must have available an amount \(\bar{x}\) of commodity at time \(T\). Purchases can be made at each of a given number of dates \(T_i\) in set \([0, T]\), with \(i \in I\) (this set is defined as in §7.2). The stage set
is \( I \). Due to operational constraints, there is a capacity \( C \) (a positive number less than or equal to \( \bar{x} \)) that limits the amount of commodity that can be purchased at each such date. The cash flow associated with the purchase of an amount of commodity \( a \) in stage \( i \) is \(- (s_i + c) a\), where \( c \) is a marginal operational cost. The holding cost \( h \) is assessed on each inventory unit available in a given stage.

The requirement on the amount of inventory to be procured by time \( T_N \equiv T \) and the limit on the amount that can be purchased at each time \( T_i \) imply that the feasible inventory set in each stage \( i \) is \( \mathcal{X}_i := [0 \vee (\bar{x} - (N - i) C), \bar{x}] \). A feasible purchase in stage \( i \) when the available inventory level is \( x \in \mathcal{X}_i \) must satisfy

\[
0 \leq a \leq C, \\
0 \vee (\bar{x} - (N - i - 1) C) \leq x + a \leq \bar{x}.
\]

These inequalities imply that the corresponding set of feasible purchases is

\[
\mathcal{A}_i(x) := [0 \vee ((0 \vee (\bar{x} - (N - i - 1) C)) - x), (\bar{x} - x) \wedge C].
\]

Like the inventory disposal asset, suppose for simplicity that the distribution of the spot price in the next stage only depends on the spot price in the current stage. The initial inventory level in stage 0 is zero. Reusing the notation used to formulate problem 7.1, the problem to be solved is

\[
\min_{\pi \in \Pi} \sum_{i=0}^{N-1} \delta_i^i \mathbb{E}^{RN} [(\bar{s}_i + c) A_i^\pi(\bar{x}_i, \bar{s}_i) + h \bar{x}_i | 0, s_0]. \tag{7.2}
\]

Like the inventory disposal asset, the inventory acquisition asset is related to the storage asset, but the requirement that a given amount of commodity be procured by a given date is critical to make the problem nontrivial. In other words, if there were no such requirement, the optimal policy would be trivially equal to doing nothing in every stage for every spot price realization. In contrast, this is not the case for the inventory disposal asset when it lacks the requirement that the entire initial inventory be sold by the end of the time horizon.

Proposition 9 states that the optimal inventory acquisition policy for problem (7.2) has a basestock target structure.
Proposition 9 (Inventory acquisition asset). The optimal policy for MDP (7.2) is of the basestock target type. In every stage $i \in I$ given a spot price $s_i \in \mathbb{R}_+$ there exists a critical inventory level $\bar{b}_i(s_i) \in \mathcal{X}_i$ such that an optimal action in state $(x_i, s_i) \in \mathcal{X}_i \times \mathbb{R}_+$ is $a^*_i(x_i, s_i) = 0 \lor ((\bar{b}_i(s_i) - x_i) \land C)$.

As with Proposition 8, the intuition behind Proposition 9 is the concavity in inventory of the optimal continuation value function of the formulation of problem (7.2) as an SDP, not shown here for brevity, in each given stage and for a given spot price, and the linearity in the amount of purchased inventory of the immediate payoff function. Moreover, the optimal basestock structure for an inventory acquisition asset shares the nontriviality and lot size characterization of the optimal policy for an inventory disposal asset.

7.4 Swing Assets

Swing contracts are widespread in energy, especially electricity and natural gas, industries. In these industries, producers or buyers often transact via contracts that specify minimum and maximum total amounts of energy to be sold or purchased at a fixed price during a given time period. Moreover, these contracts specify a maximum, and possibly a minimum, amount of energy transacted on each given trading date. Thus, a producer or a buyer that owns such a contract is obligated to deliver or purchase at least a given amount of energy, but retains the flexibility to decide how to do so during the given time period. In other words, the owners of such contracts have available a given number of “swings” and must decide how to optimally exercise them during the contract life. The distinction here is between sale-swing and purchase-swing contracts.

The existence of energy spot markets means that producers or buyers could transact in such markets at the prevailing market price, rather than through a given swing contract. Thus, the valuation and management of swing contracts involve optimally managing the gains/savings that accrue to the owner of a swing contract relative to trading in
the spot market. Consequently, the dynamics of energy spot prices are fundamental in the valuation and management of swing contracts.

**Sale-swing asset.** Consider an energy producer that has agreed to deliver at least $\bar{x}$ but no more than $\bar{\pi}$ units of energy during the time interval $[0, T]$ to a given buyer. Sales can be made at each of a given number of dates $T_i$ with $i \in I$ (the same set used in §7.2). Each sale can be made at a contractually fixed unit price equal to $K$. There is a limit $C$, a positive number, on the amount of each sale (the possibility of a minimum sale quantity per date is ignored for simplicity).

If a sale $a$ is made at time $T_i$, then the producer gains the amount $(K - s_i)a$. This is because the producer could have sold the amount of energy $a$ on the spot market for a cash flow equal to $s_ia$, but ownership of the sale-swing contract gives the producer the ability to obtain additional value relative to transacting on the spot market.\(^1\) This is a key insight that allows the problem of managing a sale-swing contract to be formulated as an MDP.

The stage set of this MDP is the set $I$. The total amount of energy sold since time 0 by time $T_i$ is denoted by $x_i$ and represents the amount of energy already sold up to stage $i$, a type of “inventory.” The feasible inventory set in stage $i$ is $X_i := [0 \lor (\bar{x} - (N - i)C), \bar{\pi}]$.

Given the feasible inventory level $x$ in stage $i$, a feasible action must satisfy the following constraints:

$$0 \leq a \leq C,$$
$$0 \lor (\bar{x} - (N - i - 1)C) \leq x + a \leq \bar{\pi}.$$  

These inequalities imply that this action must belong to the following set:

$$A_i(x) := [0 \lor (x - (0 \lor (\bar{x} - (N - i - 1)C))), (\bar{\pi} - x) \land C].$$

As in §§7.2-7.3, assume for simplicity that the current spot price is the only information available about future spot price dynamics. The

---

\(^1\) The implicit assumption here is that the operational cost of executing the sale is the same under both modes of operations.
initial inventory level in stage 0 is 0. Reusing the notation used in §§7.2-7.3, optimally managing a sale-swing asset entails solving the following problem:

\[
\max_{\pi \in \Pi} \sum_{i=0}^{N-1} \delta^i \mathbb{E}[R_N \left((K - \tilde{s}_i) A_i^\pi(\tilde{x}_i^\pi, \tilde{s}_i) - h \tilde{x}_i \mid 0, s_0\right)]. \tag{7.3}
\]

On the surface, problem (7.3) resembles problem (7.1) once the decision rule \( A_i^\pi(\tilde{x}_i^\pi, \tilde{s}_i) \) is expressed as \( -(-A_i^\pi(\tilde{x}_i^\pi, \tilde{s}_i)) \) and the selling price \( K \) is replaced with the marginal cost \( c \). Notwithstanding some remaining differences in the definition of the feasible inventory and action sets of these problems, this observation suggests that the optimal policy for a sale-swing asset is of the basestock target type. Proposition 10 demonstrates that this is indeed the case.

**Proposition 10 (Sale-swing asset).** The optimal policy for problem (7.3) is of the basestock target type. In every stage \( i \in I \) given a spot price \( s_i \) there exists a critical inventory level \( b_i(s_i) \in X_i \) such that an optimal action in state \((x_i, s_i) \in X_i \times \mathbb{R}^+\) is \( a^*_i(x_i, s_i) = 0 \lor ((x_i - b_i(s_i)) \land C)\).

Proposition 10 relies on the same intuition that underlies Propositions 8 and 9 in this chapter. Moreover, the optimal policy of a sale-swing asset shares the same complexity and lot size characterization of the optimal policies as the inventory disposal and acquisition assets (the lot size characterization holds under the assumption that the quantities \( \underline{x}, \bar{x}, \) and \( C \) are all integer multiples of some number \( Q \)).

**Purchase-swing asset.** A purchase swing-asset is the analogue of a sale-swing asset for an energy buyer. This buyer has agreed to purchase at least \( \underline{x} \) but no more than \( \bar{x} \) units of energy during the time interval \([0, T]\) from a given producer. On each of a given number of dates \( T_i \) with \( i \in I \), this buyer can purchase energy up to the limit \( C \) at a given unit price \( K \). The difference between the sale-swing asset and the purchase-swing asset is the benefit obtained by the buyer when a purchase is made. This per unit gain on a purchase made at time \( T_i \) is \((s_i - K)\), and represents the savings obtained by the buyer when
purchasing using the purchase-swing asset rather than the spot market. Replacing the term \((K - s_i)\) with this per unit gain in problem (7.3) is the only modification that needs to be made to this problem to model the optimal management of the purchase-swing asset. It follows that the analogue of Proposition 10 in this section and the properties discussed after this result hold for the purchase-swing asset.

### 7.5 Conclusions

This chapter considers the management of inventory disposal/acquisition assets and swing assets. Inventory disposal/acquisition assets are relevant for a merchant that must sell/procure a given amount of inventory by a certain date, and has the flexibility to decide when to sell/purchase this inventory. Swing assets are widespread in the energy industry and provide quantity flexibility to energy producers and users.

Variants of the basestock target structure that characterize the optimal management of storage assets are also optimal for the management of the conversion assets considered here. Moreover, these variants of the basestock target structure retain the nontriviality and the lot size characterization of the storage optimal basestock target structure.

### 7.6 Notes

The inventory disposal asset without the condition that the entire initial inventory be sold by the end of the finite horizon represents a natural resource production asset, e.g., a natural gas or petroleum reservoir or a coal mine; that is, the initial inventory represents the amount of available natural resource, e.g., oil, natural gas, or coal. In such cases the dynamics of the natural resource availability may deserve more detailed modeling. For instance, after an initial transitory period, the natural gas or petroleum that can be extracted from a reservoir in a given time period has been observed to exhibit exponential decline as the total natural gas or petroleum in the reservoir depletes. Such decline may also be characterized by stochastic behavior. Smith and McCardle [196] and Enders et al. [82], among others, model oil and
natural gas production assets as SDPs. Brown and Smith [37] take a bandit approach to the management of oil and gas production assets. Cortazar et al. [60] apply the least squares Monte Carlo approach to value a copper mine.

The inventory acquisition asset can be modified to a consumption asset, whereby the amount of acquired commodity is used as an input to a manufacturing or distribution stage. The required amount of commodity may not be known with certainty, that is, the demand for the commodity may be random, and excess supply can be carried in inventory to satisfy demand in later periods. Nascimento and Powell [163] and Secomandi and Kekre [189] discuss models in which the commodity/energy requirement is stochastic (in the model of Secomandi and Kekre [189] energy purchases can be made both in spot and forward markets incurring differential transaction costs). Kalymon [125], Gavirneni [98], Goel and Gutierrez [104, 105], Berling and Victor Martinez-de-Albéniz [13], and Kouvelis et al. [138, 139] present commodity inventory management models with stochastic demand and purchase prices (Goel and Gutierrez [105] do this in a multiechelon setting).

This chapter does not deal with cross-commodity conversion assets that involve the physical conversion of one commodity into one or more commodities. Examples include refinery assets that convert petroleum into gasoline, naphtha, and jet fuel, corn or sugarcane into ethanol, soybean into soyoil and soymeal; power plants that use natural gas; and the processing of cattle into beef products. Power plants are often subject to pollution emission caps which limit their operations. Operation of a power plant then involves choosing which days will be the most profitable – given the spark spread between power and fuel prices – to produce power and use up the available inventory of emission permits. This decision has a similar timing option structure as an inventory disposal asset. If emission permits are tradable, as with SOX in the US or carbon in Europe, then operation of a power plant is a variation on a storage option in which an inventory of emission permits can be bought, sold, and also used over time. Coal and natural gas fired power plants have also access to storage of their fuels. In the case of natural gas, in addition to underground natural gas storage facilities,
the pipeline linepack can also be used for short-term storage. Liquefied natural gas terminals also provide short-term storage opportunities. In particular, the short-term planning of sales from a liquefied natural gas regasification terminal to make space for an incoming shipment can be modeled as an inventory disposal asset.

Fleten and Kristoffersen [94], Näskäkkä and Keppo [162], and Denault et al. [71] consider hydropower generation assets. The short-term management of these assets is related to the management of inventory disposal assets. However, in this case it might not be desirable to entirely drain the reservoir that stores the water used for power generation. Modeling this aspect simply amounts to redefining the terminal inventory level to be strictly positive. Moreover, additional water inflow might occur during the planning horizon. Relevant work in the area of hydropower production, but not from a real option perspective, includes that of Drouin et al. [79] and Lamond et al. [144]. The management of windpower generation assets and liquefied natural gas production and regasification assets are closely related topics (see §5.8 in Chapter 5 for relevant references). Wallace and Fleten [215] review the related literature on energy stochastic optimization models.

Research on the management of cross-commodity conversion assets includes the work of Adkins and Paxson [2], Arvesen et al. [4], Boyabatli et al. [29], Brandão et al. [32], Devalkar et al. [74], Dockendorf and Paxson [77], Hahn and Dyer [112], Kazaz and Webster [130], Thompson et al. [207], Tseng and Barz [212], Tseng Lin [213], Wu and Chen [225], Plambeck and Taylor [168], and Thompson [206].

The swing assets considered in this chapter are based on the description of swing contracts in Geman [99, p. 294]. The insight that a swing contract can be valued and managed relative to the strategy that only operates in the spot markets appears novel. Ghiuva et al. [101], Jailliet et al. [121], Keppo [133], Barrera-Esteve et al. [8], Ross and Zhu [177], and Nadarajah et al. [160], among others, discuss the valuation and management of swing contracts.
This chapter briefly discusses a nonexhaustive list of trends for further research in the area of merchant operations. The topics considered are financial hedging in §8.1; the analysis of other commodity conversion assets in §8.2; approximate dynamic programming methods in §8.3; price model error in §8.4; endogeneity of the price process in determining an operating policy §8.5; equilibrium asset pricing in §8.6; and the choice of capacity levels in §8.7.

8.1 Financial Hedging

In addition to being the basic principle underlying risk neutral valuation in dynamically complete markets, the ability to construct dynamic portfolios of traded securities (typically, commodity and energy futures contracts) that replicate the cash flows generated by commodity conversion assets is an important risk management tool (Hull [118]). Merchants can use these replicating portfolios for financial hedging.

Specifically, shorting and periodically rebalancing a replicating portfolio amounts to eliminating the change over time of the market value of a given asset. This approach yields a financial hedging policy,
known as delta hedging (see §3.3 and expression (3.11)). Delta hedging can be used to reduce/eliminate risk capital charges that a merchant might incur in some costly states of nature.

Due to the lack of explicit representations for optimal, or near-optimal, operating policies for most commodity conversion assets, closed-form expressions for the hedges, the so called deltas, are rare. Numerical computation of the deltas is thus the norm. Monte Carlo simulation is a useful tool in this respect. Boyle et al. [30], Broadie and Glasserman [36], and Fu and Hu [97] discuss the estimation of the Greeks by Monte Carlo simulation. Glasserman [103, Chapter 7] provides a more recent review of this literature. Wang et al. [217] is a recent addition to this literature. Kaniel et al. [129], Chen and Liu [56], and Caramellino and Zanette [42] focus on the estimation of Greeks for options with multiple exercise dates.

Secomandi and Wang [191] and Secomandi et al. [190] use derivative estimation techniques to compute the deltas of natural gas transport and storage assets using term structure price models. Secomandi et al. [190] also test their approach on natural gas data. Bonnans et al. [23] compute the price sensitivities of the values of energy contracts in a stochastic programming setting. Fleten and Wallace [95] and Li and Kleindorfer [146] consider the delta hedging of hydropower generation assets and spark spread options.

More research is needed to develop delta estimation techniques under alternative price models and benchmark the performance of the resulting estimators in realistic situations. Research on hedging is particularly important in dynamically incomplete markets, for which risk neutral valuation depends on price-of-risk assumptions.

8.2 Analysis of Other Commodity Conversion Assets

This monograph examines in detail only a small number of applications: storage and inventory disposal/acquisition/swing assets. As discussed in §2.6 in Chapter 2 and in §7.6 in Chapter 7, the ideas and methods of merchant operations have much wider applicability beyond these specific applications. Important commodity conversion assets that have received substantial attention in the literature, but are ignored in
this monograph, include power plants and a variety of commodity processing assets that embed portfolios of spread/rainbow and switching options.

The structure of the optimal operating policies of these assets is typically not well understood in the literature. In particular, unlike the optimal policy of the assets that are the focus of this monograph, the optimal policy structure for these other assets is unlikely to be of the basestock type. Deepening our understanding of the structure of their optimal policies is a fruitful area for additional research. Although the computation of optimal policies is typically an intractable task, knowledge of these structures might inform the development of practical and effective methods for the computation of near-optimal policies.

8.3 Approximate Dynamic Programming

As illustrated in Chapter 6, practitioners use high-dimensional commodity price evolution models. Even though one may not want to use a full-dimensional market model of the futures curve evolution, a model with three factors would already be considered high dimensional for the exact computation of optimal operating policies. Both practitioners and scholars have thus turned to approximations that allow for the efficient computations of near-optimal operating policies.

Due to the dynamic and stochastic nature of most merchant operations problems, these approximations involve some form of approximate dynamic programming algorithm. Glasserman [103, Chapter 7] discusses the literature on the Monte Carlo pricing of American options, including approximate dynamic programming methods. This material is relevant to merchant operations problems, which tend to have an American or Bermudan structure. Moreover, as discussed in §6.8 in Chapter 6, the literature includes several approximate dynamic programming methods. Refining these methods via their application to merchant operations problems is an interesting area for additional research.

Approximate dynamic programming typically focuses on the computation of near optimal policies and lower bounds to a maximization problem. As demonstrated in Chapter 6, good upper bounds are essen-
tial to assess the performance of heuristic policies. As discussed in §6.8 in Chapter 6, there is an active stream of recent research that deals with upper bound estimation in financial and real options valuation. Continuing this line of work in the area of merchant operations, in which optimization problem are more complicated than stopping problems, is a promising area for additional research.

8.4 Price Model Error

Merchant operations relies in a fundamental manner on models of the evolution of commodity and energy prices. It is inevitable that these models are only approximations of the dynamics of these prices. In other words, these models embed errors. It is thus important to assess the impact of model errors on the merchant management of different commodity conversion assets. The impact here can be in terms of asset valuation, which merchants use to decide how much to bid to acquire an asset, the asset operating policy and, when relevant, the asset financial hedging policy, which affect the operational and financial cash flows earned by a merchant when managing an asset. When this impact is practically important, developing methods to mitigate the negative consequences of this impact is relevant. This type of research thus combines empirical and methodological aspects. Secomandi et al. [190] investigate these issues in the context of merchant commodity storage. It would be interesting to investigate the topic of price model error in the context of different commodity conversion assets.

8.5 Price Impact

A key underlying assumption behind this monograph is that a merchant’s trading decisions do not affect market prices. That is, the capacity of the commodity conversion asset under a merchant’s control is “small” relative to the market size or, equivalently, commodity markets are sufficiently liquid. In such a setting, market liquidity considerations do not play a role in optimizing the operating policy of a conversion asset, and a price-taker modeling approach is justified. However, liquidity in physical spot markets can be limited. In these cases, the price-taker
approach is not justified and may generate suboptimal operating policies once liquidity constraints are taken into account. Studying the extent of this suboptimality and developing merchant operations methods in the presence of limited liquidity are interesting avenues for additional research. Some research along these directions includes the papers of Martínez-de-Albéniz and Vendrell Simón [155], Felix et al. [89], and Chaton and Durand-Viel [54].

8.6 Equilibrium Asset Pricing

Reduced-form models like Pilipovic/Schwartz-Smith or future term structure factor models (see §4.2 and §4.3 in Chapter 4) simply take commodity price dynamics as exogenously given. As such, reduced-form models make strong assumptions about the stationarity of commodity price processes over time. In contrast, equilibrium models derive the dynamics of commodity price from the statistical properties of random environmental shocks and the joint endogenous production, storage, and consumption behavior of optimizing producers, merchants, and consumers. By relating commodity prices to deeper economic fundamentals, equilibrium models have at least some hope of being able to account for changing statistical properties of commodity prices. Section 4.4 in Chapter 4 gives pointers to recent theoretical work on the relationship between investor preferences and commodity risk premia and to recent empirical work on the macroeconomic and microeconomic drivers of commodity prices. While option valuation theory is concerned with computing market prices, merchant traders frequently trade on views about future changes in fundamentals that differ from the prevailing market view. Commodity pricing when agents have strategic market power or private information and the impact of market segmentation between physical and financial markets for commodities are relatively unexplored topics that are not well understood.

8.7 Capacity Choice

This monograph assumes that the capacity of a commodity commodity conversion asset is given. That is, a merchant has already decided
how to size the assets that it controls, e.g., how much space and injection/withdrawal capacity of a given natural gas storage asset to rent and for how long. This capacity choice problem is important in practice when a merchant faces risk capital charges, that is, when there are frictions in capital markets. In this situation, a merchant might not be able to fund all the available projects, so that a merchant might have to choose which projects to select and their size. The resulting problem might resemble a portfolio optimization problem, even though projects might be selected in a sequential manner rather than simultaneously.

It would be interesting to learn how merchants currently address this project selection and sizing problem, and to investigate whether optimization techniques might be relevant to support merchants in solving such problem. This topic is also related to the capacity choice and sequential investment problems studied in the real option literature (see, e.g., Dixit and Pindyck [76]).
Acknowledgements

We thank Yunfan Gu for offering insightful comments on Chapters 3 and 4, Selva Nadarajah for his help in producing some of the figures in Chapter 4, and Charles Corbett, the editor in chief, and an anonymous reviewer for their constructive feedback on the initial version of this monograph. We also thank Uday Karmarkar, the previous editor in chief, for his invitation to write this monograph. Chapter 5 uses by permission content from N. Secomandi, Optimal Commodity Trading with a Capacitated Storage Asset, *Management Science*, 56, 3, 2010, 449-467, Copyright 2010, The Institute for Operations Research and the Management Sciences, 7240 Parkway Drive, Suite 300, Hanover, MD 21076 USA. Chapter 6 uses by permission content from G. Lai, F. Margot, N. Secomandi, An Approximate Dynamic Programming Approach to Benchmark Practice-based Heuristics for Natural Gas Storage Valuation, *Operations Research*, 58, 3, 2010, 564-582, Copyright 2010, The Institute for Operations Research and the Management Sciences, 7240 Parkway Drive, Suite 300, Hanover, MD 21076 USA. This work was in part supported by NSF grant CMMI 1129163.
References

[1] D. Adelman. Math programming approaches to approximate dynamic pro-
multidimensional American options. Management Science, 50:1222–1234,
2004.
[8] C. Barrera-Esteve, F. Bergeret, C. Dossal, E. Gobet, A. Meziou, R. Munos,
and D. Reboul-Salze. Numerical methods for the pricing of swing options: A
stochastic control approach. Methodology And Computing In Applied Prob-
[10] D. Belomestny, C. Bender, and J. Schoenmakers. True upper bounds for
bermudan products via non-nested monte carlo. Mathematical Finance, 19:53–
71, 2009.


References


References

158 References

159 References


References


References


References


[161] S. Nadarajah, F. Margot, and N. Secomandi. Relaxations of approximate linear programs for the real option management of commodity storage. Work-
References


References


